We denote the order by \( \order \). Representations of the circle have already appeared in several areas of mathematics. Given representations of \( \hat{\Lambda} \), quivers are string algebras; the study of “continuous string algebras” remains largely unexplored territory.

\[ |S| = 0 \quad |s_0 = s_2 = s_1 = s_3| = 2 \quad |s_1 = s_2| = 4 \]

Definition. Choose an even number of points \( S \) in \( \mathbb{S}^1 \) (possibly 0). If \( S \neq \emptyset \), index the elements counterclockwise starting with \( s_0 \). Additinally, set \( s_0 = s_1 \).

1. If \( S = \emptyset \) we give \( \mathbb{S}^1 \) the cyclic counterclockwise order.
2. If \( S \neq \emptyset \) the even-indexed elements of \( S \) are sinks and the odd-indexed elements are sources, inducing a partial order on \( S^1 \).
3. We denote the order by \( \succeq \) and call this our orientation.

Example.

Definition. Given \( S \) and \( \succeq \) a continuous quiver of type \( \hat{\Lambda} \) is a category \( Q \).

The objects of \( Q \) are the points of \( S^1 \).

If \( S = \emptyset \) then \( Q \) is the category \( S^1 \).

If \( S \neq \emptyset \) we define \( Q \) as follows:

Let \( g \) be the unique morphism from \( x \) to \( y \) that travels counterclockwise less than one rotation.

If \( |S| \geq 2 \) then we allow two distinct points \( x, y \in S^1 \).

Hom\(_{Q}(x, y) = \{ g_{x,y} \circ \omega_{x} \circ \cdots \circ \omega_{y} : n \in \mathbb{N} \} = \{ \omega_{n} \circ g \circ y \circ u : n \in \mathbb{N} \} \)

Thus we may “lift” \( V \) to a representation \( \hat{M} \) of a quiver \( Q \).

Example.

Partition \( Q \)

\[ \text{String} \quad \text{Band} \]

We then examine the structure of \( \hat{M} \) and “push it down.”

Lemma (Hanson-R. [4]). A pointwise finite-dimensional representation \( V \) partitions the continuous quiver \( Q \) in finitely-many pieces. On each piece \( V \) is constant up to isomorphism.

We now have enough to understand the summands of a pf representation of \( S^1 \).

Theorem (Hanson-R. [4]). Let \( V \) and \( W \) be representations of a continuous quiver \( Q \) of type \( \hat{\kappa} \).

1. Suppose \( V \) and \( W \) are strings. Then \( V \cong W \) if and only if they lift to the same interval of \( \mathbb{R} \) modulo \( 2\pi \).
2. Suppose \( V \) and \( W \) are bands; let \( \hat{V} \) and \( \hat{W} \) be the “traveling around” maps for \( V \) and \( W \), respectively. Then \( V \cong W \) if and only if there is a matrix \( A \) such that \( \hat{V} = A \hat{W} \).
3. If \( V \) is a string and \( W \) is a band then \( V \not\cong W \).

4. Finistic Representations

- If \( |S| \geq 2 \), for each \( 0 \leq i < |S| \) let \( \Pi_i \) be the closed region on \( S^1 \) from \( s_i \) to \( s_{i+1} \). The interior of the region is denoted \( R_i \). Then \( W \) is a summand of \( V \) restricted to \( \Pi_i \) and the support of \( W \) is contained in \( R_i \).

Lemma (Hanson-R. [4]). Suppose \( W \) is a summand of \( V \) restricted to \( \Pi_i \) and the support of \( W \) is contained in \( R_i \). Then \( W \) is a summand of \( V \).

- Each \( \Pi_i \) is totally-ordered.
- Crawley-Boevey proved that representations of totally-ordered sets decompose into intervals indecomposables [2] (in our case, strings).
- Applied to each \( \Pi_i \), we have \( V \cong \bigoplus_{i=0}^{n} W_i \), where each \( W_i \) is the sum of summands of \( V \) whose support is contained in \( R_i \).

Definition. A representation \( V \) is called finistic if each \( W_i \) as described is 0.