On the global dimension of an algebra

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Motivation

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Problem

Find easy invariants to determine whether $\mathrm{gdim}\,\Lambda$ is finite or infinite.

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- vertices the non-isomorphic simple Λ-modules,
- single arrows $S \to T$ with $\mathrm{Ext}^1(S, T) \neq 0$.

No Loop Conjectures

Proposition

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Strong No Loop Conjecture (Zacharia, 1990)

If S simple with $\mathrm{pdim}(S) < \infty$, then $E(\Lambda)$ has no loop at S.

Cartan determinant

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$$\mathrm{cd}(\Lambda) = \det C(\Lambda)$$

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Cartan Determinant Conjecture (Auslander)

If $\operatorname{gdim} \Lambda < \infty$, then $\operatorname{cd}(\Lambda) = 1$.

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Remark

None of the two conjectures is established for general artin algebras.

Brief history of NLC in algebracially closed case

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Let A finite dimensional algebra over $k = \bar{k}$.

1. (Lenzing, 1969; Igusa, 1990) *NLC holds.*
2. (Skorodumov, 2010) *SNLC holds if A is representation-finite.*
3. *Other partial solutions of SNLC obtained by Burgess-Saorin, Diracca-Koenig, Green-Sølberg-Zacharia, Mamaridis-Papista, Paquette, Liu-Morin, Igusa, Zacharia.*

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1. Establish Strong No Loop Conjecture for finite dimensional algebras over an algebraically closed field (Liu, Igusa, Paquette, 2011).

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1. Establish Strong No Loop Conjecture for finite dimensional algebras over an algebraically closed field (Liu, Igusa, Paquette, 2011).
2. Establish Cartan Determinant Conjecture for quasi-stratified artin algebras (Liu, Paquette, 2006).

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1) $[\Lambda, \Lambda] = \{\sum_i (a_i b_i - b_i a_i) \mid a_i, b_i \in \Lambda\}$.

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1) $[\Lambda, \Lambda] = \{\sum_i (a_i b_i - b_i a_i) \mid a_i, b_i \in \Lambda\}$.
2) $\mathrm{HH}_0(\Lambda) = \Lambda/[\Lambda, \Lambda]$.

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1) $[\Lambda, \Lambda] = \{\sum_i (a_i b_i - b_i a_i) \mid a_i, b_i \in \Lambda\}$.
2) $\mathrm{HH}_0(\Lambda) = \Lambda/[\Lambda, \Lambda]$.
3) Say $\mathrm{HH}_0(\Lambda)$ is *radical-trivial* if $J \subseteq [\Lambda, \Lambda]$.

Trace of matrices

Definition

For $A = (a_{ij}) \in M_n(\Lambda)$, one defines

$$\mathrm{tr}(A) = (a_{11} + \cdots + a_{nn}) + [\Lambda, \Lambda] \in \mathrm{HH}_0(\Lambda).$$

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Proposition

If $A \in M_{m\times n}(\Lambda)$ and $B \in M_{n\times m}(\Lambda)$, then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Trace of endomorphisms of projective modules

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2. Given $\varphi \in \mathrm{End}_\Lambda(P)$.
3. Write $\varphi = (\varphi_{ij})_{n\times n}$, where $\varphi_{ij} \in e_i\Lambda e_j$.
4. Define
$$\mathrm{tr}(\varphi) = \mathrm{tr}\left((\varphi_{ij})_{n\times n}\right) \in \mathrm{HH}_0(\Lambda).$$

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Theorem (Lenzing)

If $\mathrm{gdim}(\Lambda) < \infty$, then $\mathrm{HH}_0(\Lambda)$ is radical-trivial.

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$\exists$ algebra morphism $p_e : \Lambda \to \Lambda_e : x \mapsto x + \Lambda(1-e)\Lambda$.

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This induces group morphism

$$H_e : \mathrm{HH}_0(\Lambda) \to \mathrm{HH}_0(\Lambda_e) : x + [\Lambda, \Lambda] \mapsto p_e(x) + [\Lambda_e, \Lambda_e].$$

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Lemma

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- If $\mathrm{idim}\, S_e < \infty$, then every M_Λ is e-bounded.

e-trace of endomorphisms of e-bounded modules

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Remark

If $\mathrm{idim}\, S_e < \infty$, then $\mathrm{tr}_e(\varphi)$ defined for any endomorphism $\varphi \in \mathrm{mod}\,\Lambda$.

Additivity of the e-trace

Lemma

- *Let* $\mathrm{mod}\Lambda$ *have commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0. \end{array}$$

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If any two of L, M, N *are e-bounded, then*

$$\mathrm{tr}_e(\varphi) = \mathrm{tr}_e(\phi) + \mathrm{tr}_e(\psi).$$

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In particular, $\mathrm{tr}_e(\varphi) = \mathrm{tr}_e(\varphi_{r+1}) = \mathrm{tr}_e(0) = 0$.

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Proof. Let $\bar{x} = x + \Lambda(1-e)\Lambda \in \mathrm{rad}(\Lambda_e)$, where $x \in \Lambda$.
Then $\bar{x} = \bar{a}$, where $a^{n+1} = 0$ for some $n \geq 0$.

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Thus $eJe/eJ^2e = 0 \Rightarrow \mathrm{Ext}^1(S_e, S_e) = 0$.
If $\mathrm{pdim}(S_e) < \infty$, then consider Λ^{o}-simple S^{o}.

Main result

Theorem

Let A fin dim alg over field k with S simple of dimension one. If $E(A)$ has a loop at S, then $\mathrm{pdim}(S) = \mathrm{idim}(S) = \infty$.

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That is, $E(A)$ no loop at S.

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Theorem (Igusa, Liu, Paquette, 2011)

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Proof. Since $\bar{k} = k$, we have $A \approx kQ/I$.

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1) I nilpotent $\Leftrightarrow$ its maximal idempotent is zero.
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Λ is *right standardly stratified* if it admits chain of ideals

$$0 = I_0 \subset \cdots \subset I_r \subset I_{r+1} = \Lambda,$$

I_{i+1}/I_i is right stratifying in Λ/I_i, $i = 0, \ldots, r$.

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The statement follows from inductive hypothesis.

Example

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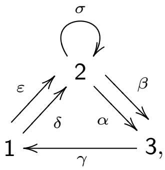

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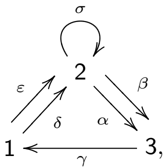

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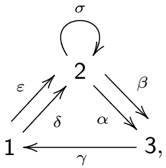

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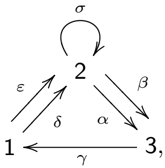
Λ quasi-stratified with quasi-stratification chain

$$\begin{aligned} 0 \quad & \subset \quad < \varepsilon > \subset < \varepsilon, \alpha > \subset < \varepsilon, \alpha, \delta > \subset < \varepsilon, \alpha, \delta, e_2 > \\ & \subset \quad < \varepsilon, \alpha, \delta, e_2, e_3 > \subset \Lambda, \end{aligned}$$

Example

Let Λ be given by



$$\sigma^2 = \sigma\beta = \beta\gamma = \gamma\delta = \varepsilon\alpha = \varepsilon\sigma = \varepsilon\beta = \delta\alpha - \delta\sigma\alpha = 0.$$

Λ neither left nor right standardly stratified.

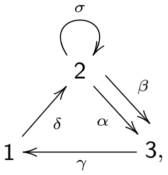
Λ quasi-stratified with quasi-stratification chain

$$\begin{aligned} 0 \quad &\subset \quad <\varepsilon> \subset <\varepsilon, \alpha> \subset <\varepsilon, \alpha, \delta> \subset <\varepsilon, \alpha, \delta, e_2> \\ &\subset \quad <\varepsilon, \alpha, \delta, e_2, e_3> \subset \Lambda, \end{aligned}$$

where first ideal left projective, others right projective over the quotient by the preceding one.

How to find quasi-stratification chain

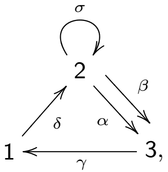
$$0 \subset < \varepsilon > \subset < \varepsilon, \alpha > \subset < \varepsilon, \alpha, \delta > .$$



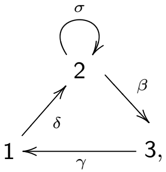
$$\sigma^2 = \sigma\beta = \beta\gamma = \gamma\delta = \delta\alpha - \delta\sigma\alpha = 0$$

How to find quasi-stratification chain

$$0 \subset < \varepsilon > \subset < \varepsilon, \alpha > \subset < \varepsilon, \alpha, \delta > .$$



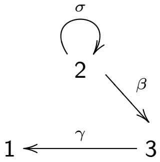
$$\sigma^2 = \sigma\beta = \beta\gamma = \gamma\delta = \delta\alpha - \delta\sigma\alpha = 0$$



$$\sigma^2 = \sigma\beta = \beta\gamma = \gamma\delta = 0.$$

How to find quasi-stratification chain

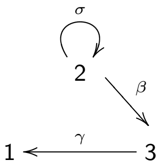
$$< \varepsilon, \alpha, \delta > \subset < \varepsilon, \alpha, \delta, e_2 > \subset < \varepsilon, \alpha, \delta, e_2, e_3 > \subset \Lambda.$$



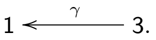
$$\sigma^2 = \sigma\beta = \beta\gamma = 0.$$

How to find quasi-stratification chain

$$< \varepsilon, \alpha, \delta > \subset < \varepsilon, \alpha, \delta, e_2 > \subset < \varepsilon, \alpha, \delta, e_2, e_3 > \subset \Lambda.$$

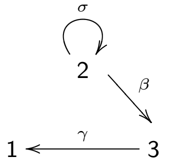


$$\sigma^2 = \sigma\beta = \beta\gamma = 0.$$



How to find quasi-stratification chain

$$< \varepsilon, \alpha, \delta > \subset < \varepsilon, \alpha, \delta, e_2 > \subset < \varepsilon, \alpha, \delta, e_2, e_3 > \subset \Lambda.$$



$$\sigma^2 = \sigma\beta = \beta\gamma = 0.$$

$$1 \xleftarrow{\gamma} 3.$$

$$1.$$