Representation theory of an infinite quiver

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October 2 - 7, 2011

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- in classification of noetherian hereditary abelian categories with Serre functor [RVDB].
- in construction of cluster categories with infinite cluster structure [HJ].

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2. To give some necessary and sufficient conditions for this category to have AR-sequences and to admit Serre functor.

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There is a Nakayama equivalence

$$\nu : \ \mathrm{proj}(Q) \ \to \ \mathrm{inj}(Q) : \ P_x \ \mapsto \ I_x.$$

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- $\mathrm{rep}^+(Q) \cap \mathrm{rep}^-(Q) = \mathrm{rep}^b(Q)$.

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- P_1 is injective in $\mathrm{rep}^+(Q)$ but $P_1 \not\cong I_x$, for every $x \in Q_0$.

AR-translation in $\mathrm{rep}(Q)$

Definition

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define $\mathrm{DTr}M \in \mathrm{rep}^-(Q)$ by short exact sequence

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2. If $M \in \mathrm{rep}^-(Q)$ with minimal injective co-resolution
$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{g} I_1 \longrightarrow 0,$$
define $\mathrm{TrD}\, M \in \mathrm{rep}^+(Q)$ by short exact sequence
$$0 \longrightarrow \nu^-(I_0) \xrightarrow{\nu^-(g)} \nu^-(I_1) \longrightarrow \mathrm{TrD}\, M \longrightarrow 0.$$

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4. *Every AR-sequence in* $\mathrm{rep}^+(Q)$ *is as stated above, and in particular, it has finite dimensional starting term.*

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- A valuation (d_{XY}, d'_{XY}) is symmetric if $d_{XY} = d'_{XY}$.
- In this case, valued arrow $X \to Y$ is replaced by $d_{X,Y}$ unvalued arrows from X to Y.
- In this way, $\Gamma_{\mathcal{A}}$ is a partially valued translation quiver in which all possible valuations are non-symmetric.

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If Δ is section of Γ, then there is embedding

$$\Gamma \to \mathbb{Z}\Delta : \tau^n x \mapsto (-n, x).$$

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Let Q infinite. For $X \in \Gamma_{\mathrm{rep}^+(Q)}$, exactly one of following occurs.

(1) $X = \tau^{-n}P_x$ *with* $x \in Q_0$ *and* $n \geq 0$, *called* preprojective;
(2) $X = \tau^{n}I_x$ *with* $x \in Q^+$ *and* $n \geq 0$, *called* preinjective;
(3) X *is neither preprojective nor preinjective, called* regular.

Proposition

Let Γ be component of $\Gamma_{\mathrm{rep}^+(Q)}$.

If one representation in Γ is preprojective (preinjective, regular), then all representations in Γ are preprojective (preinjective, regular). In this case, Γ is called preprojective (preinjective, regular).

The preprojective component

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4. *If* Q *has no right infinite path, then* $\mathcal{P}_Q \cong \mathbb{N}Q^{\mathrm{op}}$.

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4. *If Q has no right infinite path, then $\mathcal{P}_Q \cong \mathbb{N}Q^{\mathrm{op}}$.*
5. *Otherwise, $\mathcal{P}_Q$ has right-most section formed by its infinite dimensional representations, and consequently, the τ-orbits in $\mathcal{P}_Q$ are all finite.*

To construct the preprojective component

$\mathcal{P}_Q$ can be constructed from $\mathbb{N}Q^{\rm op}$ as follows.

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(1) Define $f(0,x) = \dim_k P_x \in \mathbb{N} \cup \{\infty\}$, for $x \in Q_0$.

(2) Extend f to additive function

$$f : \mathbb{N}Q^{\rm op} \to \mathbb{N} \cup \{\infty\}$$

in unique way so that if $f(u) = \infty$ and $\exists u \rightsquigarrow v$, then $f(v) = \infty$.

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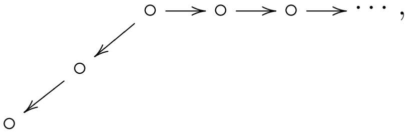
(3) $\mathcal{P}_Q \cong$ subquiver of $\mathbb{N}Q^{\rm op}$ of the (n,x) with $f(n,x) < \infty$ or $f(n,x) = \infty$ with $n = 0$ or $f(n-1,x) < \infty$.

An example

Let Q be as follows :



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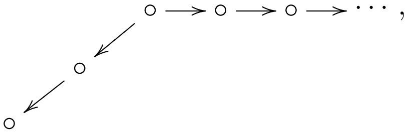
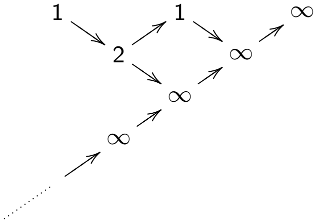


$\mathcal{P}_Q$:



Preinjective components

For $x \in Q^+$, let Q_x^+ connected component of Q^+ containing x.

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(3) *$\mathcal{I}$ contains only finite dimensional representations.*

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in unique way so that if $f(u) = \infty$ and $\exists v \rightsquigarrow u$, then $f(v) = \infty$.

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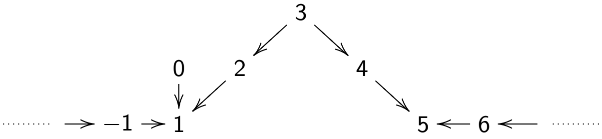
4. The preinjective components correspond bijectively to the connected components of the subquiver of $\mathbb{N}^{-}Q^{\rm op}$ of the (n,x) with $f(n,x) < \infty$.

Example

Let Q be as follows :



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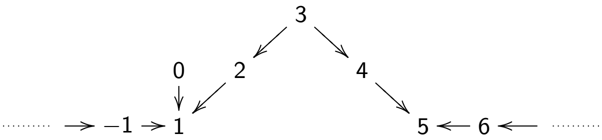
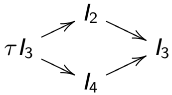


$\Gamma_{\mathrm{rep}^+(Q)}$ has two preinjective components:

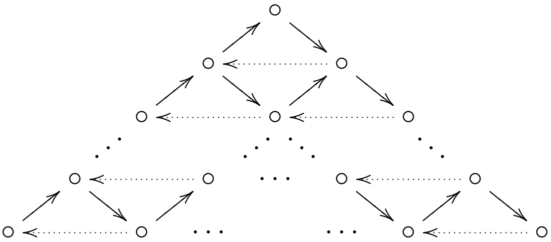
$\{I_0\}$ and



Wings

Definition

A *finite wing* is a trivially valued translation quiver as follows :



Regular components

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1. *If Γ has no infinite dimensional or pseudo-projective representation, then* $\Gamma \cong \mathbb{Z}\mathbb{A}_\infty$.

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1. *If* Γ *has no infinite dimensional or pseudo-projective representation, then* $\Gamma \cong \mathbb{Z}\mathbb{A}_\infty$.
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3. *If* Γ *has pseudo-projective but no infinite dimensional representations, then the pseudo-projective representations form a right infinite path. In particular,* $\Gamma \cong \mathbb{N}\mathbb{A}_\infty$.

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3. *If Γ has pseudo-projective but no infinite dimensional representations, then the pseudo-projective representations form a right infinite path. In particular, $\Gamma \cong \mathbb{N}\mathbb{A}_\infty$.*
4. *If Γ has both pseudo-projective representations and infinite dimensional representations, then Γ is a finite wing.*

Consequnce

Corollary

The AR-quiver $\Gamma_{\mathrm{rep}^+(Q)}$ of $\mathrm{rep}^+(Q)$ is symmetrically valued.

Infinite Dynkin diagrams

Definition

The *infinite Dynkin diagrams* are as follows :

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$\mathbb{D}_\infty: \quad \circ - \circ - \circ - \circ - \cdots$, with one more $\circ$ joined by an edge below the second vertex:

$$\begin{array}{ccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \cdots \\ & & | & & & & & & \\ & & \circ & & & & & & \end{array}$$

The non-Dynkin case

Theorem

If Q is not of finite or infinite Dynkin type, then $\Gamma_{\mathrm{rep}^+(Q)}$ has infinitely many regular components.

The infinite Dynkin case

Theorem

If Q is an infinite Dynkin quiver, then $\Gamma_{\mathrm{rep}^+(Q)}$ has at most 4 components, of which at most 1 is preinjective and at most 2 are regular.

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If Q is an infinite Dynkin quiver, then $\Gamma_{\mathrm{rep}^+(Q)}$ has at most 4 components, of which at most 1 is preinjective and at most 2 are regular.

1. *Q of type $\mathbb{A}_\infty \Rightarrow$ no regular component.*
2. *Q of type $\mathbb{D}_\infty \Rightarrow$ exactly one regular component.*

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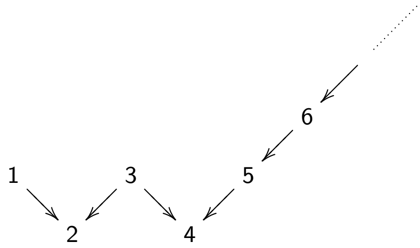
2. *Q of type $\mathbb{D}_\infty \Rightarrow$ exactly one regular component.*
3. *Q of type $\mathbb{A}_\infty^\infty \Rightarrow$ at most 2 regular components.*

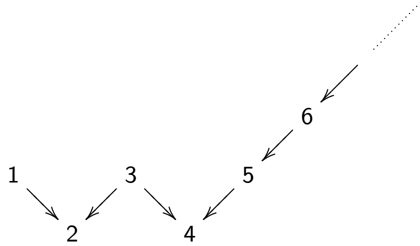
Example

Q :



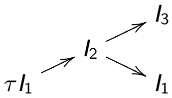
Example

Q :



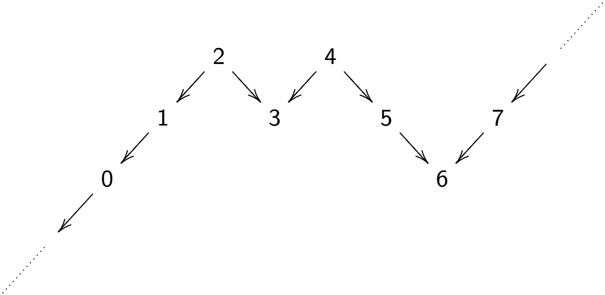
Then $\Gamma_{\mathrm{rep}^+(Q)}$ consists of two components :

$\mathcal{P}_Q \cong \mathbb{N}\mathbb{A}_\infty$ and



Example

Let Q be the following quiver of type $\mathbb{A}_{\infty}^{\infty}$:



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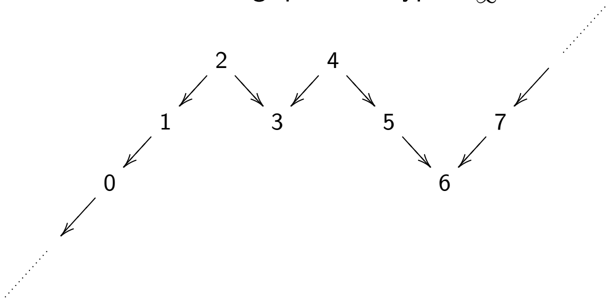
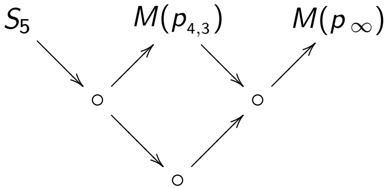


Let $p_{i,j} : i \rightsquigarrow j$, and $p_\infty : 2 \rightsquigarrow -\infty$.

Example

Then $\Gamma_{\mathrm{rep}^+(Q)}$ has a finite regular component of wing type:



Example

and a regular component of shape $\mathbb{Z}\mathbb{A}_\infty$: