Homology theory of double complexes with application to Koszul duality

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Algebra Seminar

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2. Total complex of a double complex

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$$\begin{array}{ccccc} C(\mathcal{A}) & \longrightarrow & K(\mathcal{A}) & \longrightarrow & D(\mathcal{A}) \\ \downarrow{\scriptstyle F^C} & & \downarrow{\scriptstyle F^K} & & \downarrow{\scriptstyle F^D} \\ C(\mathcal{B}) & \longrightarrow & K(\mathcal{B}) & \longrightarrow & D(\mathcal{B}), \end{array}$$

where F^C is the component-wise application of F.

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3. To pass $\mathfrak{F}^C$ to the derived categories, we need to introduce a homology theory of double complexes.

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4. Given $n \in \mathbb{Z}$, $\{M^{i,n-i} \mid i \in \mathbb{Z}\}$ is the n-*diagonal* of $M^{\cdot\cdot}$.

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5. The *vertical shift* of $M^{\cdot\cdot}$ can be defined similarly.

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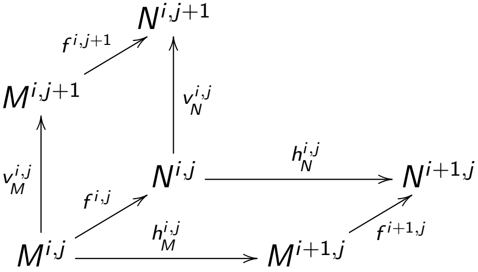

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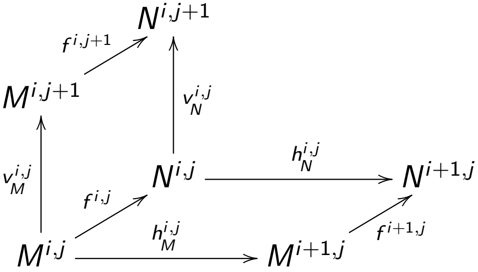

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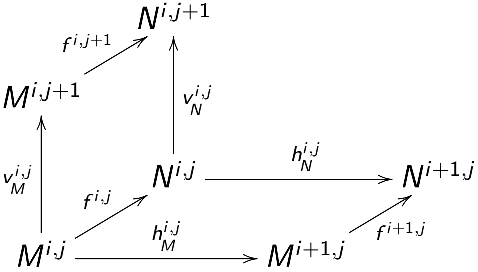

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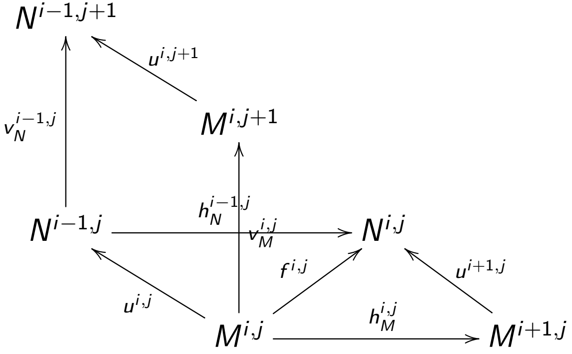

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$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
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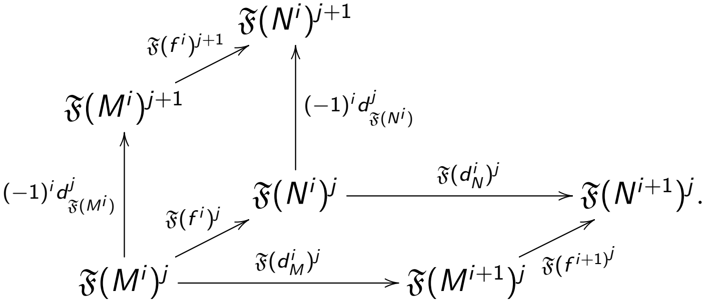

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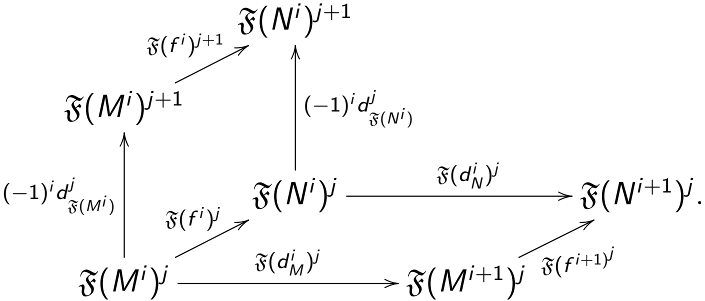

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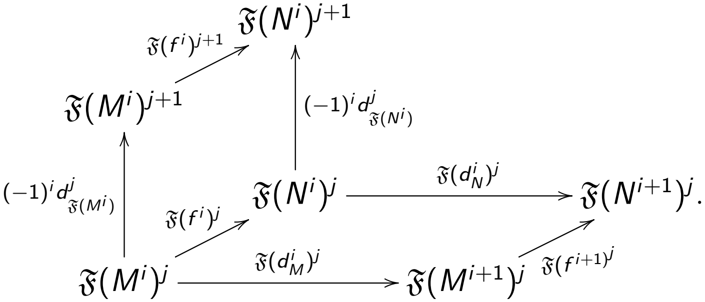

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Example

Consider $\sigma : \mathcal{A} \to C(\mathcal{A}) : M \to M[0]$. Then $\sigma^C = 1_{C(\mathcal{A})}$.

Corollary

Any exact functor $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ induces a commutative diagram

$$\begin{array}{ccc} C(\mathcal{A}) & \longrightarrow & K(\mathcal{B}) \\ {\scriptstyle \mathfrak{F}^C}\downarrow & & \downarrow{\scriptstyle \mathfrak{F}^K} \\ C(\mathcal{B}) & \longrightarrow & K(\mathcal{B}), \end{array}$$

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Proposition

If $\mathfrak{F} : \mathcal{A} \to C(\mathcal{B})$ and $\mathfrak{G} : \mathcal{B} \to C(\mathcal{C})$ are exact functors, then

$$(\mathfrak{G}^C \circ \mathfrak{F})^C = \mathfrak{G}^C \circ \mathfrak{F}^C.$$

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3. In general, $\mathfrak{F}^C$ does not send all acyclic complexes to acyclic ones.
4. We are obliged to consider special subcategories of complex category.

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Theorem

(1) *There exists a commutative diagram*

$$\begin{array}{ccccc} C^{\dagger}(\mathcal{A}) & \longrightarrow & K^{\dagger}(\mathcal{A}) & \longrightarrow & D^{\dagger}(\mathcal{A}) \\ \downarrow{\scriptstyle \mathfrak{F}^{C}} & & \downarrow{\scriptstyle \mathfrak{F}^{K}} & & \downarrow{\scriptstyle \mathfrak{F}^{D}} \\ C^{\ddagger}(\mathcal{B}) & \longrightarrow & K^{\ddagger}(\mathcal{B}) & \longrightarrow & D^{\ddagger}(\mathcal{A}), \end{array}$$

where $\mathfrak{F}^{K}, \mathfrak{F}^{D}$ *are triangle-exact.*

Passage to the derived categories

1. Let $C^\dagger(\mathcal{A})$ be an additive subcategory of $C(\mathcal{A})$ such that $\mathfrak{F}^{DC}(M^\cdot)$ is diagonally bounded-below for all $M^\cdot \in C^\dagger(\mathcal{A})$.
2. $C^\ddagger(\mathcal{B}) = \mathfrak{F}^C(C^\dagger(\mathcal{A}))$ is an additive subcategory of $C(\mathcal{B})$.

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(2) *If $\mathfrak{F}, \mathfrak{G} : \mathcal{A} \to C(\mathcal{B})$ are quasi-isomorphic, then $\mathfrak{F}^D, \mathfrak{G}^D : D(\mathcal{A}) \to D(\mathcal{B})$ are isomorphic.*

Koszul functor in the gradable setting

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   where $I_x^!$ is indecomposable injective $\Lambda^!$-module.

Special subcategories of complex categories

1. Every module $M \in \Lambda$ admits a Q-graduation

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Main Result

Theorem

If $\Lambda = kQ/R$ is Koszul, then F induces triangle equivalence

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5. $F^D \circ G^D \cong \mathbb{1}_{D^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)}$.

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5. $F^D \circ G^D \cong \mathbb{1}_{D^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)}$.
6. Similarly, $F^D \circ G^D \cong \mathbb{1}_{D^{\downarrow}_{q+1,p-1}(\mathrm{Mod}\Lambda)}$.

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Corollary

If Q has no right infinite path or no left infinite path, then

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