Homological dimensions of simple modules over artinian rings

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Motivation

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Theorem (Happel)

If Λ is artin algebra with $\operatorname{gdim}(\Lambda) < \infty$, then $D^b(\operatorname{mod}\Lambda)$ has almost split triangles.

Reduction to simple modules

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Observation

To study $\mathrm{gdim}(\Lambda)$, it suffices to consider the homological dimensions of simple modules.

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2. If $\Lambda = k[x]/<x^2>$, then

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Objective

Using combinatorial properties of $E(\Lambda)$ to determine whether $\mathrm{gdim}(\Lambda)$ is finite or infinite.

Sufficient condition for $\operatorname{gdim}(\Lambda)$ to be finite

Proposition

If $E(\Lambda)$ *has no oriented cycle, then* $\operatorname{gdim}(\Lambda) < \infty$.

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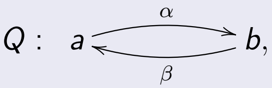
Let Λ be given by the quiver with relations:



$\alpha\beta = 0$

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Let Λ be given by the quiver with relations:

$$Q: \quad a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b, \qquad \alpha\beta = 0$$

Then $E(\Lambda) = Q$ and $\mathrm{gdim}(\Lambda) = 2$.

Conjectures

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- If $E(\Lambda)$ has a loop at some S, then $\mathrm{pdim}(S) = \infty$.
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3. The simple module k has minimal projective resolution

$$\cdots \longrightarrow \Lambda \xrightarrow{x} \Lambda \xrightarrow{x} \Lambda \longrightarrow k \longrightarrow 0.$$

Objective of this talk

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1. We shall present a proof of the SNLC (Igusa-Liu-Paquette, 2011) for finite dimensional algebras over an algebraically closed field.
2. This proof is a localization of a technique used by Lenzing (1969) to study the Hochschild homology.
3. This result of Lenzing's implies the NLC for finite dimensional algebras over an algebraically closed field (Igusa, 1990).

Notation

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$S_e = e\Lambda/eJ$, the simple Λ-module supported by e.

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Let $M \in \operatorname{mod}\Lambda$ *have minimal projective resolution*

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Corollary

$E(\Lambda)$ has a loop at $S_e \Leftrightarrow eJ^2e \neq eJe$.

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Definition

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Remark

If $a, b \in \Lambda$, then $\overline{ab} = \overline{ba}$ in $\mathrm{HH}_0(\Lambda)$.

Trace of a matrix over Λ

Definition

For $A = (a_{ij}) \in M_n(\Lambda)$, one defines

$$\mathrm{tr}(A) = (a_{11} + \cdots + a_{nn}) + [\Lambda, \Lambda] \in \mathrm{HH}_0(\Lambda).$$

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Proposition

If $A \in M_{m\times n}(\Lambda)$ and $B \in M_{n\times m}(\Lambda)$, then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

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which is independent of the decomposition of P.

Example

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Let $a \in \Lambda$. Consider the left multiplication

$$a_L : \Lambda \to \Lambda : x \mapsto ax.$$

We have

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Definition (Lenzing)

$$\mathrm{tr}(\varphi) = \sum_{i=0}^{n} (-1)^i \,\mathrm{tr}(\varphi_i) \in \mathrm{HH}_0(\Lambda).$$

Lenzing's result

Remark

If $\mathrm{gdim}(\Lambda) < \infty$, then $\mathrm{tr}(\varphi)$ is defined for every endomorphism $\varphi \in \mathrm{mod}\,\Lambda$.

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Theorem (Lenzing)

If $\mathrm{gdim}(\Lambda) < \infty$, then $\mathrm{HH}_0(\Lambda)$ is radical-trivial.

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Remark

For any $a \in \Lambda(1-e)\Lambda$, we have

$$H_e(a+[\Lambda,\Lambda]) = 0.$$

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e-bounded modules

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Remark

1. M is e-bounded $\Leftrightarrow \mathrm{Ext}^i(M, S_e) = 0$, for $i >> 0$.
2. $\mathrm{idim}\, S_e < \infty \Rightarrow$ every $M \in \mathrm{mod}\,\Lambda$ is e-bounded.

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In particular, $\mathrm{tr}_e(0) = 0$.

Additivity of the e-trace

Lemma

Let $\mathrm{mod}\Lambda$ *have comm. diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0. \end{array}$$

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If any two of L, M, N *are e-bounded, then all are e-bounded with*

$$\mathrm{tr}_e(\varphi) = \mathrm{tr}_e(\theta) + \mathrm{tr}_e(\psi).$$

e-bounded filtration

Definition

An *e-bounded filtration* for $M \in \operatorname{mod}\Lambda$ is a series

$$0 = M_{r+1} \subset M_r \subset \cdots \subset M_1 \subset M_0 = M$$

of submodules of M such that

$$M_i/M_{i+1} \text{ is } e\text{-bounded, for } i = 0, 1, \ldots, r.$$

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Remark

M admits e-bounded filtration $\Rightarrow$ M is e-bounded.

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Proof. Let $\varphi_i = \varphi|_{M_i} \Rightarrow \exists$ comm. diagram with exact rows

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Let $\varphi \in \mathrm{End}(M)$ *with e-bounded filtration*

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In particular, $\mathrm{tr}_e(\varphi) = \mathrm{tr}_e(\varphi_{r+1}) = \mathrm{tr}_e(0) = 0$.

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That is, $\bar{a} \in [\Lambda_e, \Lambda_e]$.

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As a consequence, $e\Lambda(1-e)\Lambda e \subseteq eJ^2e$.

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$\Rightarrow \delta_e(a + \Lambda(1-e)\Lambda) = 0 \Rightarrow a = eae \in eJ^2e.$

Thus, $eJe = eJ^2e \Rightarrow \mathrm{Ext}^1(S_e, S_e) = 0.$

Application to finite dimensional algebras

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Proof. Let $S = S_e \Rightarrow e$ is basic with $e\Lambda e = ke + eJe$.

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- $\mathrm{idim}(S) < \infty \Rightarrow \mathrm{Ext}^1(S, S) = 0$.
- $\mathrm{pdim}(S) < \infty \Rightarrow D(S) \in \mathrm{mod}\Lambda^{\mathrm{op}}$ is 1-dimensional with $\mathrm{idim}D(S) < \infty \Rightarrow E(\Lambda^{\mathrm{op}})$ has no loop at $D(S)$.

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Theorem (Igusa, Liu, Paquette, 2011)

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Proof. We may assume that Λ is basic.

Then every simple Λ-module is one dimensional.

Status quo for artinian rings

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2. More advanced technique is needed, for instance, AR-theory in $D^b(\mathrm{mod}\Lambda)$.

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1. *If* $\mathrm{Ext}^1(S,S) \neq 0$, *then* $\mathrm{Ext}^n(S,S) \neq 0$ *for infinitely many* n.
2. *Let* S *have a minimal projective resolution*

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0.$$

If P_0 *is direct summand of* P_1, *then* P_0 *is direct summand of* P_n *for infinitely many* n.