The radical nipotency of the module category of a hereditary algebra of Dynkin type

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• A: connected non-simple artin algebra.

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- **2** mod *A*: category of finitely generated left *A*-modules.

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• rad(mod A): Jacobson radical of mod A.

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Theorem (Auslander)

A representation-finite $\Leftrightarrow (\operatorname{rad}(\operatorname{mod} A))^m = 0$ for some $m \ge 1$.

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Theorem (Auslander)

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Objective

Understand the representation theory of representation-finite artin algebras in terms of the nilpotency of rad(mod A).

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Problem

• Given a class of representation-finite algebras A,

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Problem

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- Given m > 0, find all A with rad(modA) of nilpotency m and study their representation theory.

In this talk, we calculate the nilpotency of rad(modA) in case A is hereditary.

• Let A be hereditary artin algebra.

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- projective covers P_1, \ldots, P_n ;
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- **3** Write $\underline{\dim} M = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, for $M \in \text{mod} A$,

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with d_i multiplicity of S_i as composition factor of M.

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- $\exists ! \text{ isomorphism } \Phi_A : \mathbb{Z}^n \to \mathbb{Z}^n : \underline{\dim} P_i \mapsto -\underline{\dim} I_i$ called the Coxeter transformation of $K_0(A)$.

The *Ext-quiver* Q_A of artin algebra A is a valued quiver:

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The *Ext-quiver* Q_A of artin algebra A is a valued quiver:

• the vertices are the non isomorphic simple A-modules.

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$$\operatorname{Ext}^{1}_{A}(S, T) \neq 0$$

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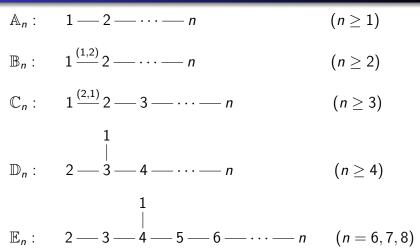
•
$$d = \dim_{\operatorname{End}(\mathcal{T})}\operatorname{Ext}^1_A(S, \mathcal{T})$$

• $d' = \dim \operatorname{Ext}^1_A(S, T)_{\operatorname{End}(S)}$.

Dynkin diagram

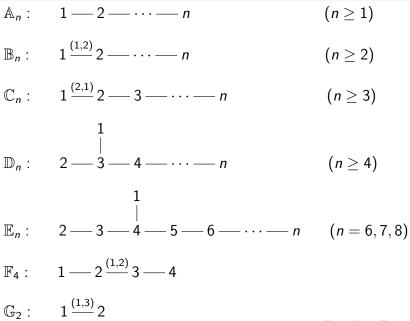
$$A_n:$$
 $1 - 2 - \cdots - n$
 $(n \ge 1)$
 $\mathbb{B}_n:$
 $1 \frac{(1,2)}{2} 2 - \cdots - n$
 $(n \ge 2)$
 $\mathbb{C}_n:$
 $1 \frac{(2,1)}{2} 2 - 3 - \cdots - n$
 $(n \ge 3)$
 $\mathbb{D}_n:$
 $2 - 3 - 4 - \cdots - n$
 $(n \ge 4)$

Dynkin diagram



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Dynkin diagram



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• \overline{Q}_A is a Dynkin diagram.

Let A be hereditary of finite representation type.

- \overline{Q}_A is a Dynkin diagram.
- **2** Φ_A is of finite order c_A , called the *Coxeter order* for *A*.

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				\mathbb{D}_n					
C _A	n+1	2 <i>n</i>	2 <i>n</i>	2(n-1)	12	18	30	12	6

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Theorem (Liu, Todorov, 2023)

If A is hereditary artin algebra of finite representation type,

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Theorem (Liu, Todorov, 2023)

If A is hereditary artin algebra of finite representation type, then rad(modA) is of nilpotency $c_A - 1$.

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Let $f : X \to Y$ be non-zero map in mod A.



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• Then $f \in \operatorname{rad}^{s}(X, Y) \setminus \operatorname{rad}^{s+1}(X, Y)$ for some $s \ge 0$.

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• Put dp(f) = s, called the *depth* of f.

Let $f: X \to Y$ be non-zero map in modA.

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Observation

If dp(f) = s > 0 with X and Y indecomposable,

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Observation

If dp(f) = s > 0 with X and Y indecomposable, then AR-quiver Γ_A contains a path $X \rightsquigarrow Y$ of length s.

• Let *A* be of finite representation type.

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Objective Define the *depth* of modA by

- Let A be of finite representation type.
- Objective Define the *depth* of modA by

 $dp(modA) = max\{dp(f) \mid f \text{ non-zero maps in } modA\}.$

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- Let A be of finite representation type.
- Define the *depth* of modA by
 dp(modA) = max{dp(f) | f non-zero maps in modA}.
- Then, dp(modA) + 1 is the nilpotency of rad(modA).

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• For each simple module $S \in \text{mod}A$, we fix

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- Define the *depth* of modA by
 dp(modA) = max{dp(f) | f non-zero maps in modA}.
- Then, dp(modA) + 1 is the nilpotency of rad(modA).

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- For each simple module $S \in \text{mod}A$, we fix
 - a projective cover $\pi_{\!\scriptscriptstyle S}: P_{\!\scriptscriptstyle S} o S;$

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- For each simple module $S \in \text{mod}A$, we fix
 - a projective cover $\pi_{\!\scriptscriptstyle S}: P_{\!\scriptscriptstyle S} o S$;
 - an injective envelope $\iota_{s}:S\rightarrow\textit{I}_{s}.$

- Let A be of finite representation type.
- Define the *depth* of modA by
 dp(modA) = max{dp(f) | f non-zero maps in modA}.
- Then, dp(modA) + 1 is the nilpotency of rad(modA).
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 - a projective cover $\pi_{\!\scriptscriptstyle S}: P_{\!\scriptscriptstyle S} o S$;
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Theorem (Chaio, Liu, 2012)

 $dp(modA) = max\{dp(\iota_s \circ \pi_s) \mid S \in modA \text{ is simple}\}.$



Construct translation quiver $\mathbb{Z}\Delta$ by knitting \mathbb{Z} copies of Δ .

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Proposition

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Assume that $\overline{\Delta}$ is a tree. Given any vertices a, b in $\mathbb{Z}\Delta$, all $a \rightsquigarrow b$ in $\mathbb{Z}\Delta$ have the same length, written as d(a, b).

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• Let Γ be translation quiver with translation τ .



1 Let Γ be translation quiver with translation τ .

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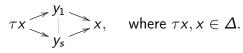
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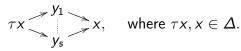
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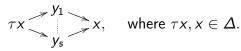


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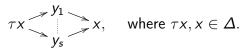
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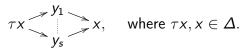
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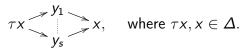


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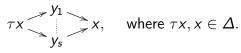
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- If Γ contains a section Δ, then it embeds in ZΔ as a convex translation subquiver.

Characterization of representation-finite hereditary algebras

Theorem (Liu, Yin, 2022)



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 $S \in \operatorname{mod} A \text{ is simple} \Rightarrow P_s[2] = \tau^{-c_A} P_s.$

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• rad(mod A) is of nilpotency $c_A - 1$.