ANOTHER CHARACTERIZATION OF TILTED ALGEBRAS

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Abstract. We give a new characterization of tilted algebras by the existence of certain special subquivers in their Auslander-Reiten quiver. This result includes the existent characterizations of this kind and yields a way to obtain more tilted quotient algebras from a given algebra.

Introduction

Since its introduction by Happel and Ringel, the theory of tilted algebras has been one of the most important topics in the representation theory of artin algebras; see, for example, [7, 12, 17, 19, 20]. Indeed, this class of algebras is closely related to hereditary algebras, a relatively well-understood class of algebras. Initially, tilted algebras are characterized by the existence of a slice in their module category, or equivalently, the existence of a slice module; see [19, (4.2)]. Later, the convexity of a slice module was replaced by a weaker condition; see [5]. Most recently, a slice module is replaced by a sincere module which is not the middle term of any short chain in the module category; see [9].

All the above-mentioned characterizations of tilted algebras require some knowledge of the entire module category, and hence, they are rather difficult to be verified for algebras of infinite representation type. To overcome this difficulty, replacing the convexity of a slice with respect to arbitrary maps by the convexity with respect to irreducible maps, one characterizes tilted algebras by the existence of an Auslander-Reiten component which contains a faithful section admitting no backward maps to its Auslander-Reiten translate; see [13, 22]. Later, a section in this characterization is replaced by a slightly weaker notion of a left section; see [1].

Observe that a section, as well as a left section, requires some knowledge of an entire Auslander-Reiten component. In this paper, by relaxing the convexity of a slice in the module category, we obtain a locally defined notion of a *cut* in the Auslander-Reiten quiver. Our main result says that an artin algebra is tilted if and only if its Auslander-Reiten quiver contains a faithful cut admitting no backward map to its Auslander-Reiten translate. This not only is easy to be verified but also yields a way to obtain more tilted quotient algebras from a given artin algebra.

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1. Preliminaries

Throughout this paper, A stands for an artin algebra. We shall denote by $\mathrm{mod}A$ the category of finitely generated left A-modules, by $\mathrm{ind}A$ the full subcategory of $\mathrm{mod}A$ generated by the indecomposable modules. Recall that a module T in $\mathrm{mod}A$ is called *tilting* if $\mathrm{pdim}(T) \le 1$, $\mathrm{Ext}^1_A(T,T) = 0$, and the number of non-isomorphic indecomposable direct summands of T is equal to the number of non-isomorphic simple A-modules. The *Jacobson radical* of $\mathrm{mod}A$, written as $\mathrm{rad}(\mathrm{mod}A)$, is the ideal generated by the non-invertible maps between indecomposable modules, while the *infinite radical* $\mathrm{rad}^\infty(\mathrm{mod}A)$ is the intersection of all powers $\mathrm{rad}^n(\mathrm{mod}A)$ with $n \ge 0$. A map in $\mathrm{rad}(\mathrm{mod}A)$ will be called a *radical* map.

We shall use freely some standard terminology and some basic results of the Auslander-Reiten theory of irreducible maps and almost split sequences, for which we refer to [3]. As usual, we shall denote by $\varGamma_A$ the Auslander-Reiten quiver of $\mathrm{mod}A$. This is a translation quiver whose vertex set is a complete set of representatives of the isomorphism classes of modules in $\mathrm{ind}A$, and whose arrows correspond to irreducible maps, and whose translation is given by the Auslander-Reiten translations $\tau = \mathrm{DTr}$ and $\tau^- = \mathrm{TrD}$, where D denotes the standard duality between $\mathrm{mod}A$ and $\mathrm{mod}A^{\mathrm{op}}$. For simplicity, we shall write $\tau X = 0$ if X is projective and $\tau^- X = 0$ if X is injective.

Let $\varSigma$ be a full subquiver of $\varGamma_A$. The *annihilator* of $\varSigma$, written as $\mathrm{ann}(\varSigma)$, is the intersection of all annihilators $\mathrm{ann}(M)$ with $M \in \varSigma$. One says that $\varSigma$ is *faithful* if $\mathrm{ann}(\varSigma) = 0$ and *sincere* if every simple A-module is a composition factor of some module in $\varSigma$. Recall that $\varSigma$ is a *section* in a connected component $\varGamma$ of $\varGamma_A$ if $\varSigma$ is a connected subquiver of $\varGamma$, which contains no oriented cycle, meets each τ-obit in $\varGamma$ exactly once, and is convex in $\varGamma$, that is, every path in $\varGamma$ with end-points belonging to $\varSigma$ lies entirely in $\varSigma$; see [14, (2.1)].

Recall that a *path* in $\mathrm{ind}A$ is a sequence of non-zero radical maps

$$X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

between modules in $\mathrm{ind}A$; and such a path is called *non-zero* if $f_n \cdots f_1 \neq 0$. The following result is implicitly included in the proof of [13, (1.2)].

1.1. Lemma. *Let X, Y be modules in $\mathrm{ind}A$ such that $\mathrm{rad}^\infty(X,Y) \neq 0$. For each integer $n > 0$, $\mathrm{ind}A$ contains a non-zero path*

$$X \xrightarrow{u_n} Y_n \xrightarrow{f_n} Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow{f_1} Y,$$

where $u_n \in \mathrm{rad}^\infty(X, Y_n)$ and $f_n, \cdots, f_1$ are irreducible; and a non-zero path

$$X \xrightarrow{g_1} X_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \xrightarrow{g_n} X_n \xrightarrow{v_n} Y,$$

where $g_1, \cdots, g_n$ are irreducible and $v_n \in \mathrm{rad}^\infty(X_n, Y)$.

1.2. Definition. Let $f : X \to Y$ be a map in $\mathrm{mod}A$. We define the *depth* of f, written as $\mathrm{dp}(f)$, to be infinity in case $f \in \mathrm{rad}^\infty(X,Y)$; and otherwise, to be the integer $n \ge 0$ for which $f \in \mathrm{rad}^n(X,Y)$ but $f \notin \mathrm{rad}^{n+1}(X,Y)$.

REMARK. A map $f : X \to Y$ in ind$A$ is irreducible if and only if $\mathrm{dp}(f) = 1$.

The following result is well known.

1.3. LEMMA. *Let $f : X \to Y$ be a radical map in* ind$A$. *If* $\mathrm{dp}(f) < \infty$, *then*

$$f = f_1 + \cdots + f_n + g,$$

where $g \in \mathrm{rad}^\infty(X, Y)$, and $f_1, \ldots, f_n$ are non-zero composites of irreducible maps between modules in ind$A$.

A path $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$ in Γ_A is *sectional* if $X_{i-1} \neq \tau X_{i+1}$ for every $0 < i < n$; *presectional* if there exists an irreducible map $u_i : \tau X_{i+1} \oplus X_{i-1} \to X_i$ for each $0 < i < n$; and *non-zero* if there exist irreducible maps $f_i : X_{i-1} \to X_i$, $i = 1, \ldots, n$, such that $f_n \cdots f_1 \neq 0$. It is well known that a sectional path is presectional. For convenience, we quote the following result from [14, (1.15)].

1.4. LEMMA. *If $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$ is a presectional path in Γ_A, then there exist irreducible maps $f_i : X_{i-1} \to X_i$, $i = 1, \ldots, n$, such that* $\mathrm{dp}(f_n \cdots f_1) = n$.

Applying first the Harada-Sai Lemma; see [8] and then Lemma 1.4, we obtain immediately the following result.

1.5. LEMMA. *If Σ is a finite subquiver of Γ_A, then the lengths of the non-zero paths in* ind$A$ *passing through only modules in Σ are bounded, and consequently, Σ has no infinite sectional path.*

The weak convexity defined below is essential for our investigation.

1.6. DEFINITION. Let Σ be a full subquiver of Γ_A. We shall say that Σ is

(1) *convex* in ind$A$ if every path in ind$A$ with end-points in Σ passes through only modules which are isomorphic to modules in Σ;

(2) *weakly convex* in ind$A$ if every non-zero path in ind$A$ with end-points in Σ passes through only modules which are isomorphic to modules in Σ.

1.7. LEMMA. *Let Σ be a finite subquiver of Γ_A, which is weakly convex in* ind$A$.

(1) *If $X, Y \in \Sigma$, then* $\mathrm{rad}^\infty(X, Y) = 0$.

(2) *The endomorphism algebra of the direct sum of the modules in Σ is connected if and only if Σ is connected.*

Proof. (1) Suppose that there exists a non-zero map $f \in \mathrm{rad}^\infty(X, Y)$, for some $X, Y \in \Sigma$. For each integer $n > 0$, by Lemma 1.1, ind$A$ has a non-zero path

$$X \xrightarrow{g_n} Y_n \xrightarrow{f_n} Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow{f_1} Y,$$

where $f_1, \ldots, f_n$ are irreducible. Since Σ is weakly convex, $Y_1, \ldots, Y_n \in \Sigma$, a contradiction to Lemma 1.5.

(2) Suppose that the endomorphism algebra of the direct sum of the modules in Σ is connected. Let $X, Y \in \Sigma$ be distinct modules. Then Σ contains modules $X = X_1, X_2, \ldots, X_n = Y$ such that, for each $1 \leq i < n$, there exists a non-zero map f_i from X_i to X_{i+1} or from X_{i+1} to X_i. By Statement (1), $\mathrm{dp}(f_i) < \infty$; and by Lemma 1.3, Γ_A has a non-zero path ρ_i from X_i to X_{i+1} or from X_{i+1} to X_i, for

all $1 \leq i < n$. Since Σ is weakly convex, all the paths ρ_i with $1 \leq i < n$ lie in Σ. This shows that Σ is connected. The proof of the lemma is completed.

Recall that A is *tilted* if $A = \mathrm{End}_H(T)$, where H is a hereditary artin algebra and T is a tilting module in mod$H$; see [7]. It is a well-known result of Ringel's that A is tilted if and only if mod$A$ contains a slice; see [19, (4.2)]. Observe that a slice in mod$A$ is precisely the full additive subcategory of mod$A$ generated by the modules isomorphic to those in a *slice* of Γ_A as defined below.

1.8. Definition [19]. A full subquiver Δ of Γ_A is called a *slice* if it satisfies the following conditions.

(1) The subquiver Δ is sincere and convex in ind$A$.
(2) If $X \in \Delta$, then $\tau X \notin \Delta$.
(3) If $X \to Y$ is an arrow in Γ_A with $Y \in \Delta$, then either X or $\tau^- X$ belongs to Δ.

For convenience, we reformulate Ringel's result as follows. Although it is stated in [19, (4.2)] for a finite dimensional algebra over an algebraically closed field, the same proof work for an artin algebra.

1.9. Theorem [19]. *Let A be an artin algebra, and let Δ be a full subquiver of Γ_A.*

(1) *If $A = \mathrm{End}_H(T)$ with H hereditary and T a tilting H-module, then T determines a slice in Γ_A generated by the direct summands of* $\mathrm{Hom}_A(T, D(H))$.
(2) *The subquiver Δ is a slice if and only if $S = \oplus_{X\in\Delta} X$ is a tilting module in* mod$A$ *such that $H = \mathrm{End}_A(S)$ is hereditary. In this case, $D(S_H)$ is a tilting H-module such that $A = \mathrm{End}_H(D(S))$ and Δ is the slice determined by $D(S)$.*

Remark. In case A is connected, in view of Lemma 1.7, we see that a slice Δ of Γ_A is necessarily connected. As a consequence, Δ is contained in a connected component $\mathcal{C}$ of Γ_A, which is actually a section in $\mathcal{C}$; see [7, (7.1)] and [19, (4.2)]. Such a connected component of Γ_A is called a *connecting component*.

2. Main results

In this section, we shall present our main results, that is to characterize tilted algebras in terms of the notion of a cut as defined below and to show how to obtain some tilted quotient algebras from a given algebra.

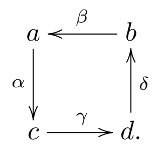
2.1. Definition. A full subquiver Δ of Γ_A is called a *cut* if, for each arrow $X \to Y$, the following conditions are verified.

(1) If $X \in \Delta$, then either Y or τY, but not both, belongs to Δ.
(2) If $Y \in \Delta$, then either X or $\tau^- X$, but not both, belongs to Δ.

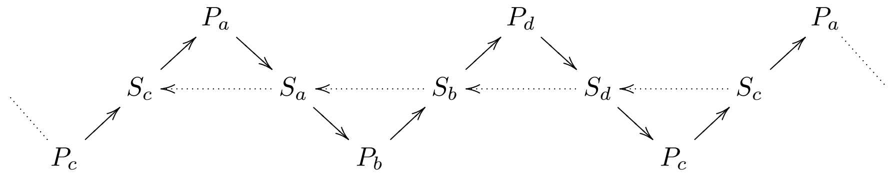
Remark. It is known that a section in a connected component of Γ_A is a cut; see [16, (2.2)]. For this reason, a cut is called a *presection* in [2, (1.3)].

Example. (1) If Γ is a connected component of Γ_A which is a non-homogeneous stable tube, then each ray or coray is a cut. It is evident that a stable tube contains no section.

(2) Let A be an algebra with radical squared zero given by the quiver



Its Auslander-Reiten quiver Γ_A is as follows:



where P_a, P_b, P_c, P_d are the indecomposable projective modules, and S_a, S_b, S_c, S_d are the simple modules. We see that $\Delta : P_b \longrightarrow S_b \longrightarrow P_d$ is a cut in Γ_A. Observe that Δ does not meet the τ-orbit of P_a or the τ-orbit of P_c.

The following result exhibits the relation between a slice and a cut.

2.2. Lemma. *A full subquiver of Γ_A is a slice if and only if it is a cut which is sincere and convex in* ind$A$.

Proof. Let Δ be a full subquiver of Γ_A. Suppose that Δ is a sincere cut which is convex in ind$A$. In particular, Δ satisfies the condition stated in Definition 1.8(3). Let X be a module in Δ. If τX also belongs to Δ, then Γ_A admits a path $\tau X \to Y \to X$. By the convexity of Δ, we have $Y \in \Delta$, a contradiction to Definition 2.1(1). Therefore, Δ is a slice.

Conversely, suppose that Δ is a slice. In particular, Δ satisfies the condition stated in Definition 2.1(2). Let $X \to Y$ be an arrow in Γ_A with $X \in \Delta$. If Y is not projective, then $\tau Y \to X$ is an arrow in Γ_A, and by Definition 1.8(3), either τY or Y belongs to Δ. Otherwise, since Δ is sincere, there exists a non-zero map $f : Y \to M$ for some $M \in \Delta$. By the convexity of Δ in ind$A$, we have $Y \in \Delta$. This shows that Δ also satisfies the condition stated in 2.1(1). The proof of the lemma is completed.

Let Γ be a connected component of Γ_A. We say that Γ is *semi-regular* if it contains no projective module or no injective module. In case Γ contains no oriented cycle, one says that Γ is *preprojective* (respectively, *preinjective*) if every τ-orbit in Γ contains a projective (respectively, injective) module. The following result is not explicitly stated in any existent literature; compare [10, (1.1)(i), (1.2)(i)].

2.3. Proposition. *Let A be an artin algebra, and let $\mathcal{C}$ be a sincere preprojective or preinjective component of Γ_A. If $\mathcal{C}$ is semi-regular, then A is tilted with $\mathcal{C}$ being a connecting component.*

Proof. We shall only consider the case where $\mathcal{C}$ is preinjective without projective modules. Then, $\mathcal{C}$ contains a section Δ; see [14, (2.4)]. In particular, Δ is a finite cut of Γ_A. Let $f : X \to Y$ be a non-zero map with $X \in \mathcal{C}$ and $Y \in \Gamma_A$. Since $\mathcal{C}$ has no oriented cycle and X has only finitely many successors in $\mathcal{C}$, we deduce from Lemma 1.7(2) that $\mathrm{dp}(f) < \infty$. By Lemma 1.3, ind$A$ has a non-zero path $X \rightsquigarrow Y$

of irreducible maps, and hence, $Y \in \mathcal{C}$. Making use of this fact and the convexity of $\varDelta$ in $\mathcal{C}$, we see that $\varDelta$ is convex in $\text{ind}A$.

Now, let I be an injective module in $\varGamma_A$. We claim that $\text{Hom}_A(M, I) \neq 0$ for some $M \in \varDelta$. Indeed, we may assume that $I \notin \varDelta$. Since $\mathcal{C}$ is sincere, there exists a non-zero map $g : X \to I$ with $X \in \mathcal{C}$. As shown above, $\text{ind}A$ has a path of irreducible maps

$$X = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \longrightarrow X_{r-1} \xrightarrow{f_r} X_r = I$$

with $X_i \in \mathcal{C}$ such that $f_r \cdots f_1 \neq 0$. Since $\varDelta$ is a section, $X_i = \tau^{n_i} M_i$, where $M_i \in \varDelta$ and $n_i \in \mathbb{Z}$ with $1 \geq n_i - n_{i+1} \geq 0$; see [14, (2.3)]. Since X_r is injective and not in $\varDelta$, we have $n_r < 0$. If $n_0 > 0$, then $n_s = 0$ for some $1 \leq s \leq r$, and our claim follows. Suppose that $n_0 < 0$. Since $\mathcal{C}$ has no projective module, every minimal right almost split map for an module in $\mathcal{C}$ is surjective. Using this, we obtain an infinite path of irreducible maps

$$\cdots \longrightarrow Y_t \xrightarrow{h_t} Y_{t-1} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow{h_1} Y_0 = X$$

such that $gh_1 \cdots h_t \neq 0$ for every $t \geq 1$. Since $\varDelta$ is a finite section, some of the Y_t lies in $\varDelta$. This establishes our claim. By Lemma 2.2, $\varDelta$ is a slice of $\varGamma_A$. The proof of the proposition is completed.

2.4. LEMMA. *Let $\varDelta$ be a finite cut of $\varGamma_A$, which is weakly convex in* $\text{ind}A$*.*

(1) *If $X \in \varDelta$, then neither τX nor $\tau^- X$ belongs to $\varDelta$.*

(2) *If $X, Y \in \varDelta$, then* $\text{Hom}_A(X, \tau Y) = 0$ *and* $\text{Hom}_A(\tau^- X, Y) = 0$.

Proof. (1) Let $X \in \varDelta$. Suppose that $\tau X \in \varDelta$. Consider an almost split sequence

$$0 \longrightarrow \tau X \xrightarrow{(f_1, u_1)} Y_1 \oplus Z_1 \xrightarrow{\binom{g_1}{v_1}} X \longrightarrow 0,$$

where $Y_1 \in \varGamma_A$. By the condition stated in Definition 2.1(1), $Y_1 \notin \varDelta$. Since $\varDelta$ is weakly convex, $g_1 f_1 = 0$, and consequently, $Z_1 = 0$; see the corollary of [11, (1.3)]. Using again Definition 2.1(1), we see that $\tau Y_1 \in \varDelta$. This yields a sectional path $Y_1 \to X$ in $\varGamma_A$ and an irreducible monomorphism $f_1 : \tau X \to Y_1$ in $\text{mod}A$, where $\tau X \in \varDelta$ and $Y_1 \notin \varDelta$. Assume, for some $n \geq 1$, that there exists a sectional path

$$\rho_n : \ Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = X$$

in $\varGamma_A$ and an irreducible monomorphism $f_n : \tau Y_{n-1} \to Y_n$ in $\text{mod}A$, where $\tau Y_{n-1}, \ldots, \tau Y_0 \in \varDelta$, while $Y_n, \ldots, Y_1 \notin \varDelta$. By Definition 2.1(1), $\tau Y_n \in \varDelta$. Since f_n is an irreducible monomorphism, there exists an almost split sequence

$$0 \longrightarrow \tau Y_n \xrightarrow{\begin{pmatrix} f_{n+1} \\ u \\ h \end{pmatrix}} Y_{n+1} \oplus Z_{n+1} \oplus \tau Y_{n-1} \xrightarrow{(g, v, f_n)} Y_n \longrightarrow 0,$$

where $Y_{n+1} \in \varGamma_A$ and $f_{n+1} : \tau Y_n \to Y_{n+1}$ is a monomorphism. Observe that

$$\rho_{n+1} : \ Y_{n+1} \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = X$$

is a presectional path in $\varGamma_A$. Since $Y_n \notin \varDelta$ and $Y_0 \in \varDelta$, by Lemma 1.4 and the weak convexity of $\varDelta$, $Y_{n+1} \notin \varDelta$. In particular, $Y_{n+1} \neq \tau Y_{n-1}$, and thus, ρ_{n+1} is a sectional path in $\varGamma_A$. By induction, we obtain an infinite sectional path

$$\cdots \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = X$$

in Γ_A such that $\tau Y_i \in \Delta$, for all $i \geq 0$. This yields an infinite sectional path

$$\cdots \longrightarrow \tau Y_n \longrightarrow \tau Y_{n-1} \longrightarrow \cdots \longrightarrow \tau Y_1 \longrightarrow \tau Y_0$$

in Δ, a contradiction to Lemma 1.5. Thus, $\tau X \notin \Delta$, and consequently, $\tau^- X \notin \Delta$.

(2) We shall prove only the first part of Statement (2). Suppose on the contrary that there exists a non-zero map $f_0 : X \to \tau Y_0$, where $X, Y_0 \in \Delta$. By Statement (1), $\tau Y_0 \notin \Delta$. Consider an almost split sequence

$$0 \longrightarrow \tau Y_0 \xrightarrow{g} Z \xrightarrow{h} Y_0 \longrightarrow 0.$$

Since g is a monomorphism, we may find an irreducible map $g_1 : \tau Y_0 \to Z_1$ with $Z_1 \in \Gamma_A$ such that $g_1 f_0 \neq 0$. Since Δ is weakly convex, $Z_1 \notin \Delta$. Then, by Definition 2.1(2), $Y_1 = \tau^- Z_1 \in \Delta$. Continuing this process, we obtain an infinite path of irreducible maps

$$\tau Y_0 \xrightarrow{g_1} \tau Y_1 \longrightarrow \cdots \longrightarrow \tau Y_{n-1} \xrightarrow{g_n} \tau Y_n \longrightarrow \cdots,$$

with $g_n \cdots g_1 f_0 \neq 0$, where $Y_i \in \Delta$ and $\tau Y_i \notin \Delta$, for every $i \geq 0$. Since Δ is finite, we have a contradiction to Lemma 1.5. The proof of the lemma is completed.

Using the following result, one can easily check whether a finite cut of Γ_A is weakly convex in $\operatorname{ind} A$.

2.5. PROPOSITION. *Let Δ be a cut of Γ_A. The following conditions are equivalent.*
(1) *The cut Δ is finite and weakly convex in* $\operatorname{ind} A$.
(2) $\operatorname{Hom}_A(X, \tau Y) = 0$, *for all* $X, Y \in \Delta$.
(3) $\operatorname{Hom}_A(\tau^- X, Y) = 0$, *for all* $X, Y \in \Delta$.
In this case, moreover, Δ contains no oriented cycle.

Proof. Suppose first that Δ is finite and weakly convex in $\operatorname{ind} A$. By Lemma 2.4, Statements (2) and (3) hold. Furthermore, assume that Δ has an oriented cycle

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{s-1} \longrightarrow X_s = X_0.$$

Setting $X_{s+1} = X_1$, we have $X_{i-1} = \tau X_{i+1}$ for some $1 \leq i \leq s$; see [4]. That is, $X_{i+1},\ \tau X_{i+1} \in \Delta$, a contradiction to Lemma 2.4(1).

Conversely, assume that $\operatorname{Hom}_A(X, \tau Y) = 0$, for all $X, Y \in \Delta$. It is well known that Δ is finite; see [21, Lemma 2]. Suppose that there exist $L, N \in \Delta$ such that $\operatorname{rad}^\infty(L, N) \neq 0$. Given any $n > 0$, by Lemma 1.1, $\operatorname{ind} A$ has a non-zero path

$$L \xrightarrow{u_n} N_n \xrightarrow{f_n} N_{n-1} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{f_1} N,$$

where $N_1, \ldots, N_n \in \Gamma_A$ and $f_1, \ldots, f_n$ are irreducible. If $N_1 \notin \Delta$ then, by Definition 2.1(2), $Y = \tau^- N_1 \in \Delta$, which is absurd since $\operatorname{Hom}_A(L, \tau Y) = 0$. Thus, $N_1 \in \Delta$. By induction, $N_1, \ldots, N_n \in \Delta$, a contradiction to Lemma 1.5. This shows that $\operatorname{rad}^\infty(X, Y) = 0$, for all $X, Y \in \Delta$. Now, let

$$X_0 \xrightarrow{g_1} X_1 \longrightarrow \cdots \longrightarrow X_{s-1} \xrightarrow{g_s} X_s$$

be a non-zero path ind$A$, where $X_0, X_s \in \Delta$. Since $\text{rad}^\infty(X_0, X_s) = 0$, we have $\text{dp}(g_s \cdots g_1) < \infty$, and consequently, $\text{dp}(g_i) < \infty$, for $i = 1, \ldots, s$. Applying Lemma 1.3, we obtain a non-zero path of irreducible maps

$$X_0 = Y_0 \xrightarrow{h_1} Y_1 \longrightarrow \cdots \longrightarrow Y_{t-1} \xrightarrow{h_t} Y_t = X_s$$

with $\{X_1, \ldots, X_{s-1}\} \subseteq \{Y_1, \ldots, Y_{t-1}\} \subseteq \Gamma_A$. If $Y_{t-1} \notin \Delta$, then $Z = \tau^- Y_{t-1} \in \Delta$. This yields $0 \neq h_{t-1} \cdots h_1 \in \text{Hom}_A(Y_0, \tau Z)$, a contradiction. Therefore, $Y_{t-1} \in \Delta$. By induction, $Y_1, \ldots, Y_{t-1} \in \Delta$. In particular, $X_1, \ldots, X_{s-1} \in \Delta$. That is, Δ is weakly convex in ind$A$. Similarly, we may show that Statement (3) implies Statement (1). The proof of the proposition is completed.

Now, we are ready to state our main result, which generalizes the result stated in [1, (3.7)], [13, (1.6)], [19, (4.2)] and [22, Theorem 3]. The proof is a refinement of the argument given in [13, (1.6)].

2.6. Theorem. *Let A be an artin algebra. Then A is tilted if and only if Γ_A contains a faithful cut Δ such that* $\text{Hom}_A(X, \tau Y) = 0$ *for all* $X, Y \in \Delta$; *and in this case, Δ is a slice in Γ_A.*

Proof. If A is tilted then, by Theorem 1.9(2), Γ_A contains a finite slice Δ such that the direct of its modules is a tilting module. Since tilting modules are faithful, Δ is faithful. By Lemma 2.2, Δ is a cut which is convex in ind$A$, and by Proposition 2.5, $\text{Hom}_A(X, \tau Y) = 0$ for all $X, Y \in \Delta$.

Conversely, let Δ be a faithful cut of Γ_A such that $\text{Hom}_A(X, \tau Y) = 0$ for all $X, Y \in \Delta$. By Proposition 2.5, the direct sum T of the modules in Δ is a faithful module in mod$A$ such that $\text{Hom}_A(T, \tau T) = 0$ and $\text{Hom}_A(\tau^- T, T) = 0$. In particular, $\text{Ext}^1_A(T, T) = 0$; see, for example, [3, (4.6)]. Moreover, since T is faithful, $\text{pdim}(T) \leq 1$; see [18, (1.5)].

Let X be a module in ind$A$, but not in Δ, such that $\text{Hom}_A(T, X) \neq 0$. Suppose that $\text{Hom}_A(\tau^- T, X) = 0$. Choose a non-zero map $f_0 : T_0 \to X$ with $T_0 \in \Delta$. Not being an isomorphism, f_0 factorizes through a minimal left almost split map $g : T_0 \to L$. Therefore, there exists an irreducible map $g_1 : T_0 \to T_1$ with $T_1 \in \Gamma_A$ and a map $f_1 : T_1 \to X$ such that $f_1 g_1 \neq 0$. Since $\text{Hom}_A(\tau^- T, X) = 0$, we have $\tau T_1 \notin \Delta$, and hence, $T_1 \in \Delta$. By induction, we can find an infinite path of irreducible maps

$$T_0 \xrightarrow{g_1} T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \xrightarrow{g_n} T_n \longrightarrow \cdots$$

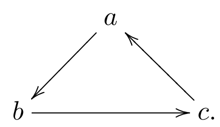
with $T_n \in \Delta$ and maps $f_n : T_n \to X$ such that $f_n g_n \cdots g_1 \neq 0$, for every $n \geq 1$. This is contrary to Lemma 1.5. Therefore, $\text{Hom}_A(\tau^- T, X) \neq 0$. As a consequence, T is a tilting module; see [18, (1.6)].

Let $H = \text{End}_A(T)$. We claim that H is hereditary. Indeed, T determines a torsion theory $(\mathscr{F}, \mathscr{T})$ in mod$A$ with torsion class $\mathscr{T}$, and a torsion theory $(\mathscr{Y}, \mathscr{X})$ in mod$H$ with torsion-free class $\mathscr{Y}$. By the Butler-Brenner Theorem; see [6], $\text{Hom}_A(T, -)$ induces an equivalence from $\mathscr{T}$ to $\mathscr{Y}$. Let P be an indecomposable projective module in mod$H$ and $v : U \to P$ be a monomorphism. Then, there exists a map $u : Q \to U$ in mod$H$, where Q is indecomposable and projective, such that $vu \neq 0$. Observing that u, v are in $\mathscr{Y}$, we may assume that $\mathscr{T}$ has morphisms $f : M \to Z$ and $h : Z \to N$, where $M, N \in \Delta$ and $Z \in$ ind$A$, such that $Q = \text{Hom}_A(T, M)$, $U = \text{Hom}_A(T, Z)$, $P = \text{Hom}_A(T, N)$, and $vu = \text{Hom}_A(T, hf)$.

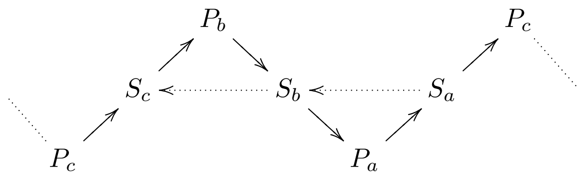
Since $vu \neq 0$, we have $hf \neq 0$. By Proposition 2.5, Δ is weakly convex in ind$A$. Thus, $Z \in \Delta$, and hence, U is projective. This establishes our claim. By Theorem 1.9(2), A is a tilted algebra and Δ is the slice determined by $D(T_H)$. The proof of the theorem is completed.

Remark. As shown by the example below, the faithfulness of the cut in Theorem 2.6 cannot be replaced by the sincereness.

Example. Let A be an algebra with radical squared zero given by the quiver



Its Auslander-Reiten quiver Γ_A is as follows:



where P_a, P_b, P_c are the indecomposable projective modules, and S_a, S_b, S_c are the simple modules. It is easy to see that $\Delta : P_b \longrightarrow S_b \longrightarrow P_a$ is a sincere cut in Γ_A such that $\mathrm{Hom}_A(X, \tau Y) = 0$ for all $X, Y \in \Delta$. However, A is not tilted.

If $B = A/I$ with I a two-sided ideal in A, then we shall identify mod$B$ as the full subcategory of mod$A$ of the modules annihilated by I. In particular, the vertices in Γ_B will be the modules in Γ_A annihilated by I, and the translation for Γ_B will be written as τ_B. The following lemma and its dual are needed for our purpose.

2.7. Lemma. *Let Δ be a cut of Γ_A such that* $\mathrm{Hom}_A(X, \tau Y) = 0$ *for all* $X, Y \in \Delta$. *Set* $B = A/\mathrm{ann}(\Delta)$ *and fix a module M in Δ.*

(1) *A minimal right almost split map $f : L \to M$ in* mod$A$ *is a minimal right almost split map in* mod$B$.
(2) *If M is projective in* mod$A$*, then it is projective in* mod$B$; *and otherwise,* $\tau_B M = \tau M$.

Proof. Firstly, we claim that Statement (2) is a consequence of Statement (1). Indeed, if M is projective in mod$A$ then, by Statement (1), the inclusion map $j : \mathrm{rad}\, M \to M$ is a minimal right almost split monomorphism in mod$B$, and thus, M is projective in mod$B$. Otherwise, mod$A$ has an almost split sequence

$$0 \longrightarrow \tau M \xrightarrow{g} L \xrightarrow{f} M \longrightarrow 0.$$

By Statement (1), L lies mod$B$, and since g is a monomorphism, so does τM. Therefore, the above sequence is an almost split sequence in mod$B$. In particular, $\tau_B M = \tau M$. This establishes our claim.

For proving Statement (1), it suffices to show, for each arrow $N \to M$ in Γ_A, that N lies in Γ_B. Indeed, by Proposition 2.5, Δ is finite and has no oriented cycle. Thus, the number of paths in Δ starting with M is finite, and the maximal length of such paths is written as d_M. If $d_M = 0$, then M is a sink in Δ. By Definition

2.1(2), $N \in \Delta$, and in particular, $N \in \Gamma_B$. Assume that $d_M > 0$. If $N \in \Delta$, then $N \in \Gamma_B$. Otherwise, $Z = \tau^- N \in \Delta$ with $d_Z < d_M$. By the induction hypothesis, $N = \tau_B Z \in \Gamma_B$. The proof of the lemma is completed.

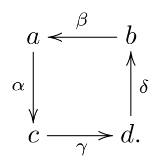
Applying Theorem 2.6, we obtain the following result, which strictly includes the corresponding results stated in [1, (3.5)], [13, (2.2)], [15, (2.7)] and [23, (3.1),(3.2)].

2.8. Theorem. *Let A be an artin algebra, and let Δ be a cut of Γ_A such that $\mathrm{Hom}_A(X, \tau Y) = 0$ for all $X, Y \in \Delta$. Then $B = A/\mathrm{ann}(\Delta)$ is a tilted algebra with Δ being a slice of Γ_B.*

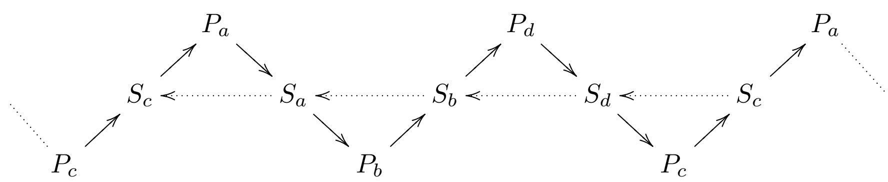
Proof. First of all, Δ is a faithful full subquiver of Γ_B. By Lemma 2.7(2), $\mathrm{Hom}_B(X, \tau_B Y) = 0$ for all $X, Y \in \Delta$. Let $M \to N$ be an arrow in Γ_B with $N \in \Delta$ and $M \notin \Delta$. By Lemma 2.7(1), $M \to N$ is an arrow in Γ_A, and by Definition 2.1(2), $\tau^- M \in \Delta$. By the dual of Lemma 2.7(2), $\tau_B^- M = \tau^- M$. This shows that Δ, as a subquiver of Γ_B, satisfies the condition stated in Definition 2.1(2), and dually, it satisfies the condition stated in Definition 2.1(1). That is, Δ is a cut of Γ_B. By Theorem 2.6, B is a tilted algebra with Δ being a slice of Γ_B. The proof of the theorem is completed.

Remark. Let A be a cluster-tilted algebra. If Σ is a local slice in Γ_A, by Theorem 19 stated in [2], $A/\mathrm{ann}(\Sigma)$ is a tilted algebra. This fact can also be deduced from Theorem 2.8. Indeed, in this situation, Σ is a cut such that $\mathrm{Hom}_A(X, \tau Y) = 0$ for all $X, Y \in \Sigma$; see [2, Lemma 9].

Example. We consider again the algebra A with radical squared zero given by the following quiver



The Auslander-Reiten quiver Γ_A is as follows:



We have seen that $\Delta : P_b \longrightarrow S_b \longrightarrow P_d$ is a cut in Γ_A. Now, it is easy to verify that $\mathrm{Hom}_A(X, \tau Y) = 0$ for all $X, Y \in \Delta$. Observe that $\mathrm{ann}(\Delta) = Ae_cA$ and $B = A/\mathrm{ann}(\Delta)$ is given by the quiver

$$d \xrightarrow{\ \delta\ } b \xrightarrow{\ \beta\ } a$$

with relation $\beta\delta$. Clearly, B is tilted of type $\mathbb{A}_3$. Note that Δ is neither a left section nor a local slice. Therefore, none of the corresponding results stated in [1, (3.5)], [13, (2.2)], [15, (2.7)] and [23, (3.1),(3.2)] is applicable in this example.

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