A PROOF OF THE STRONG NO LOOP CONJECTURE

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Abstract. The strong no loop conjecture states that a simple module of finite projective dimension over an artin algebra has no non-zero self-extension. The main result of this paper establishes this well known conjecture for finite dimensional algebras over an algebraically closed field.

Introduction

Let Λ be an artin algebra, and denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules. It is an important problem in the representation theory of algebras to determine whether Λ has finite or infinite global dimension, and more specifically, whether a simple Λ-module has finite or infinite projective dimension. For instance, the derived category $D^b(\operatorname{mod}\Lambda)$ has Auslander-Reiten triangles if and only if Λ has finite global dimension; see [7, 8]. One approach to this problem is to consider the extension quiver of Λ, which has vertices given by a complete set of non-isomorphic simple Λ-modules and single arrows $S \to T$, where S and T are vertices such that $\operatorname{Ext}^1_\Lambda(S,T)$ does not vanish. Then the *no loop conjecture* affirms that the extension quiver of Λ contains no loop if Λ is of finite global dimension, while the *strong no loop conjecture*, which is due to Zacharia, strengthens this to state that a vertex in the extension quiver admits no loop if it has finite projective dimension; see [1, 10].

The no loop conjecture was first explicitly established for artin algebras of global dimension two; see [5]. For finite dimensional elementary algebras (that is, the simple modules are all one dimensional), as shown in [10], this can be easily derived from an earlier result of Lenzing on Hochschild homology in [13]. Lenzing's technique was to extend the notion of the trace of endomorphisms of projective modules, defined by Hattori and Stallings in [9, 18], to endomorphisms of modules over a noetherian ring with finite global dimension, and apply it to a particular kind of filtration for the regular module.

In contrast, up to now, the strong no loop conjecture has only been verified for some special classes of algebras such as monomial algebras; see [2, 10], special biserial algebras; see [14], and algebras with at most two simple modules and radical cubed zero; see [12]. Many other partial results can be found in [3, 4, 6, 15, 16, 20]. Most recently, Skorodumov generalized and localized Lenzing's filtration to indecomposable projective modules. This allowed him to prove this conjecture for finite dimensional elementary algebras of finite representation type; see [17].

In this paper, we shall localize Lenzing's trace function to endomorphisms of modules in $\operatorname{mod}\Lambda$ with an e-bounded projective resolution, where e is an idempotent in Λ. The key point is that every module in $\operatorname{mod}\Lambda$ has an e-bounded projective resolution whenever the semi-simple module supported by e is of finite injective dimension. This enables us to obtain a local version of Lenzing's result.

As a consequence, we shall solve the strong no loop conjecture for a large class of artin algebras including finite dimensional elementary algebras over any field, and particularly, for finite dimensional algebras over an algebraically closed field.

1. Localized trace and Hochschild homology

Throughout, J will stand for the Jacobson radical of Λ. The additive subgroup of Λ generated by the elements $ab - ba$ with $a, b \in \Lambda$ is called the *commutator group* of Λ and written as $[\Lambda, \Lambda]$. One defines then the Hochschild homology group $\mathrm{HH}_0(\Lambda)$ to be $\Lambda/[\Lambda, \Lambda]$. We shall say that $\mathrm{HH}_0(\Lambda)$ is *radical-trivial* if $J \subseteq [\Lambda, \Lambda]$.

To start with, we recall the notion of the trace of an endomorphism φ of a projective module P in $\mathrm{mod}\Lambda$, as defined by Hattori and Stallings in [9, 18]; see also [10, 13]. Write $P = e_1\Lambda \oplus \cdots \oplus e_r\Lambda$, where the e_i are primitive idempotents in Λ. Then $\varphi = (a_{ij})_{r\times r}$, where $a_{ij} \in e_i\Lambda e_j$. The *trace* of φ is defined to be

$$\mathrm{tr}(\varphi) = \textstyle\sum_{i=1}^{r} a_{ii} + [\Lambda, \Lambda] \in \mathrm{HH}_0(\Lambda).$$

We shall collect some well known properties of this trace function in the following proposition, in which Statement (2) is precisely the reason for defining the trace to be an element in $\mathrm{HH}_0(\Lambda)$.

1.1. Proposition (Hattori-Stallings). *Let P, P' be projective modules in* $\mathrm{mod}\Lambda$*.*

(1) *If $\varphi, \psi \in \mathrm{End}_\Lambda(P)$, then $\mathrm{tr}(\varphi + \psi) = \mathrm{tr}(\varphi) + \mathrm{tr}(\psi)$.*
(2) *If $\varphi : P \to P'$ and $\psi : P' \to P$ are Λ-linear, then $\mathrm{tr}(\varphi\psi) = \mathrm{tr}(\psi\varphi)$.*
(3) *If $\varphi = (\varphi_{ij})_{2\times 2} : P \oplus P' \to P \oplus P'$, then $\mathrm{tr}(\varphi) = \mathrm{tr}(\varphi_{11}) + \mathrm{tr}(\varphi_{22})$.*
(4) *If $\psi : P \to P'$ is an isomorphism and $\varphi \in \mathrm{End}_\Lambda(P)$, then $\mathrm{tr}(\psi\varphi\psi^{-1}) = \mathrm{tr}(\varphi)$.*
(5) *If $\varphi : \Lambda \to \Lambda$ is the left multiplication by $a \in \Lambda$, then $\mathrm{tr}(\varphi) = a + [\Lambda, \Lambda]$.*

Next, we recall Lenzing's extension of this function to endomorphisms of modules of finite projective dimension. For $M \in \mathrm{mod}\Lambda$, let $\mathcal{P}_M$ denote a projective resolution of M in $\mathrm{mod}\Lambda$ as follows:

$$\cdots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

For each $\varphi \in \mathrm{End}_\Lambda(M)$, one can construct a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r, "d_i"] \arrow[d, "\varphi_i"] & P_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & P_i \arrow[r, "d_i"] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

in $\mathrm{mod}\Lambda$. We shall call $\{\varphi_i\}_{i\ge 0}$ a *lifting* of φ to $\mathcal{P}_M$. If M is of finite projective dimension, then one may assume that $\mathcal{P}_M$ is bounded and define the *trace* of φ by

$$\mathrm{tr}(\varphi) = \textstyle\sum_{i=0}^{\infty} (-1)^i\, \mathrm{tr}(\varphi_i) \in \mathrm{HH}_0(\Lambda),$$

which is independent of the choice of $\mathcal{P}_M$ and $\{\varphi_i\}$; see [13], and also [10].

Our strategy is to localize this construction. Let e be an idempotent in Λ. Set

$$\Lambda_e = \Lambda/\Lambda(1-e)\Lambda.$$

The canonical algebra projection $\Lambda \to \Lambda_e$ induces a group homomorphism

$$H_e : \mathrm{HH}_0(\Lambda) \to \mathrm{HH}_0(\Lambda_e).$$

For an endomorphism φ of a projective module in $\mathrm{mod}\Lambda$, we define its *e-trace* by

$$\mathrm{tr}_e(\varphi) = H_e(\mathrm{tr}(\varphi)) \in \mathrm{HH}_0(\Lambda_e).$$

It is evident that this e-trace function has the properties (1) to (4) stated in Proposition 1.1. We shall state another important property in the following result. For doing so, we recall that the *top* of a module in $\mathrm{mod}\Lambda$ is the quotient of the module by its Jacobson radical.

1.2. LEMMA. *Let e be an idempotent in Λ, and let P be a projective module in* $\mathrm{mod}\Lambda$ *whose top is annihilated by e. If $\varphi \in \mathrm{End}_\Lambda(P)$, then $\mathrm{tr}_e(\varphi) = 0$.*

Proof. We may assume that P is non-zero. Then $1 - e = e_1 + \cdots + e_r$, where the e_i are pairwise orthogonal primitive idempotents in Λ. Let $\varphi \in \mathrm{End}_\Lambda(P)$. By Proposition 1.1(3), we may assume that P is indecomposable. Then $P \cong e_s\Lambda$ for some $1 \le s \le r$. By Proposition 1.1(4), we may assume that $P = e_s\Lambda$. Then φ is the left multiplication by some $a \in e_s\Lambda e_s$. By Proposition 1.1(5),

$$\mathrm{tr}_e(\varphi) = H_e(a + [\Lambda, \Lambda]) = \bar{a} + [\Lambda_e, \Lambda_e],$$

where $\bar{a} = a + \Lambda(1-e)\Lambda$. Since $a = e_s a e_s = (1-e)a(1-e) \in \Lambda(1-e)\Lambda$, we get $\mathrm{tr}_e(\varphi) = 0$. The proof of the lemma is completed.

In order to extend the e-trace function, we shall say that a projective resolution $\mathcal{P}_M$ of a module M in $\mathrm{mod}\Lambda$ is *e-bounded* if all but finitely many tops of the terms in $\mathcal{P}_M$ are annihilated by e. In this case, if φ is an endomorphism of M with a lifting $\{\varphi_i\}_{i\ge 0}$ to $\mathcal{P}_M$, then it follows from Lemma 1.2 that $\mathrm{tr}_e(\varphi_i) = 0$ for all but finitely many i. This allows us to define the *e-trace* of φ by

$$\mathrm{tr}_e(\varphi) = \textstyle\sum_{i=0}^{\infty} (-1)^i \, \mathrm{tr}_e(\varphi_i) \in \mathrm{HH}_0(\Lambda_e).$$

1.3. LEMMA. *Let e be an idempotent in Λ. Then the e-trace is well defined for endomorphisms of modules in* $\mathrm{mod}\Lambda$ *having an e-bounded projective resolution.*

Proof. Let M be a module in $\mathrm{mod}\Lambda$ having an e-bounded projective resolution

$$\mathcal{P}_M : \qquad \cdots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Fix $\varphi \in \mathrm{End}_\Lambda(M)$. We first show that $\mathrm{tr}_e(\varphi)$ is independent of the choice of a lifting of φ to $\mathcal{P}_M$. By Proposition 1.1(1), it suffices to prove, for any lifting $\{\psi_i\}_{i\ge 0}$ of the zero endomorphism of M, that $\sum_{i=0}^{\infty} (-1)^i \, \mathrm{tr}_e(\psi_i) = 0$. Indeed, let $h_i : P_i \to P_{i+1}$ be morphisms such that $\psi_0 = d_1 h_0$ and $\psi_i = d_{i+1}h_i + h_{i-1}d_i$. By Proposition 1.1, $\mathrm{tr}_e(\psi_i) = \mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(h_{i-1}d_i) = \mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(d_i h_{i-1})$, for all $i \ge 1$.

On the other hand, by assumption, there exists some $m \ge 0$ such that the top of P_i is annihilated by e, for every $i \ge m$. By Lemma 1.2, $\mathrm{tr}_e(d_{m+1}h_m) = 0$ and $\mathrm{tr}_e(\psi_i) = 0$ for all $i \ge m$. This yields

$$\begin{aligned}
\textstyle\sum_{i=0}^{\infty} (-1)^i \, \mathrm{tr}_e(\psi_i) &= \mathrm{tr}_e(\psi_0) + \textstyle\sum_{i=1}^{m} (-1)^i \, \mathrm{tr}_e(\psi_i) \\
&= \mathrm{tr}_e(d_1 h_0) + \textstyle\sum_{i=1}^{m} (-1)^i \left(\mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(d_i h_{i-1})\right) \\
&= (-1)^m \, \mathrm{tr}_e(d_{m+1}h_m) \\
&= 0.
\end{aligned}$$

Next, we verify that $\mathrm{tr}_e(\varphi)$ is independent of the choice of the e-bounded projective resolution $\mathcal{P}_M$. Suppose that M has another e-bounded projective resolution

$$\mathcal{P}'_M: \qquad \cdots \longrightarrow P'_i \xrightarrow{d'_i} P'_{i-1} \longrightarrow \cdots \longrightarrow P'_0 \xrightarrow{d'_0} M \longrightarrow 0.$$

Considering φ, we get morphisms $u_i : P_i \to P'_i$ with $i \geq 0$ such that $d'_0 u_0 = \varphi d_0$ and $d'_i u_i = u_{i-1} d_i$ for $i \geq 1$. Similarly, considering the identity map 1_M, we obtain maps $v_i : P'_i \to P_i$ with $i \geq 0$ such that $d_0 v_0 = d'_0$ and $d_i v_i = v_{i-1} d'_i$ for $i \geq 1$. Observe that $\{v_i u_i\}_{i\geq 0}$ and $\{u_i v_i\}_{i\geq 0}$ are liftings of φ to $\mathcal{P}_M$ and $\mathcal{P}'_M$, respectively. By Proposition 1.1(2),

$$\textstyle\sum_{i=0}^{\infty}(-1)^i \,\mathrm{tr}_e(u_i v_i) = \sum_{i=0}^{\infty}(-1)^i \,\mathrm{tr}_e(v_i u_i).$$

The proof of the lemma is completed.

The following result says that the e-trace function is additive in some generalized Grothendieck group defined in [10].

1.4. Proposition. *Let e be an idempotent in Λ. Consider a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "u"] \arrow[d, "\varphi_L"] & M \arrow[r, "v"] \arrow[d, "\varphi_M"] & N \arrow[r] \arrow[d, "\varphi_N"] & 0 \\ 0 \arrow[r] & L \arrow[r, "u"] & M \arrow[r, "v"] & N \arrow[r] & 0 \end{tikzcd}$$

in modΛ *with exact rows. If L, N have e-bounded projective resolutions, then so does M and* $\mathrm{tr}_e(\varphi_M) = \mathrm{tr}_e(\varphi_L) + \mathrm{tr}_e(\varphi_N)$.

Proof. Assume that L and N have e-bounded projective resolutions as follows:

$$\mathcal{P}_L: \qquad \cdots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} L \longrightarrow 0$$

and

$$\mathcal{P}_N: \qquad \cdots \longrightarrow P'_i \xrightarrow{d'_i} P'_{i-1} \longrightarrow \cdots \longrightarrow P'_0 \xrightarrow{d'_0} N \longrightarrow 0.$$

By the Horseshoe lemma, there exists in modΛ a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_i \arrow[r, "d_i"] \arrow[d, "q_i"] & P_{i-1} \arrow[r] \arrow[d, "q_{i-1}"] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] \arrow[d, "q_0"] & L \arrow[r] \arrow[d, "u"] & 0 \\ \cdots \arrow[r] & P_i \oplus P'_i \arrow[r, "d''_i"] \arrow[d, "p_i"] & P_{i-1} \oplus P'_{i-1} \arrow[r] \arrow[d, "p_{i-1}"] & \cdots \arrow[r] & P_0 \oplus P'_0 \arrow[r, "d''_0"] \arrow[d, "p_0"] & M \arrow[r] \arrow[d, "v"] & 0 \\ \cdots \arrow[r] & P'_i \arrow[r, "d'_i"] & P'_{i-1} \arrow[r] & \cdots \arrow[r] & P'_0 \arrow[r, "d'_0"] & N \arrow[r] & 0 \end{tikzcd}$$

with exact rows, where $q_i = \binom{1}{0}$, $p_i = (0, 1)$ for all $i \geq 0$. In particular, the middle row is an e-bounded projective resolution of M which we denote by $\mathcal{P}_M$. Choose a lifting $\{f_i\}_{i\geq 0}$ of φ_L to $\mathcal{P}_L$ and a lifting $\{g_i\}_{i\geq 0}$ of φ_N to $\mathcal{P}_N$. It is well known; see, for example, [19, p. 46] that there exists a lifting $\{h_i\}_{i\geq 0}$ of φ_M to $\mathcal{P}_M$ such that

$$\begin{tikzcd} 0 \arrow[r] & P_i \arrow[r, "q_i"] \arrow[d, "f_i"] & P_i \oplus P'_i \arrow[r, "p_i"] \arrow[d, "h_i"] & P'_i \arrow[r] \arrow[d, "g_i"] & 0 \\ 0 \arrow[r] & P_i \arrow[r, "q_i"] & P_i \oplus P'_i \arrow[r, "p_i"] & P'_i \arrow[r] & 0 \end{tikzcd}$$

is commutative, for every $i \geq 0$. Since $h_i q_i = q_i f_i$ and $g_i p_i = p_i h_i$, we may choose to write h_i as a (2×2)-matrix whose diagonal entries are f_i and g_i. By Proposition 1.1(3), $\mathrm{tr}_e(h_i) = \mathrm{tr}_e(f_i) + \mathrm{tr}_e(g_i)$. Hence, $\mathrm{tr}_e(\varphi_M) = \mathrm{tr}_e(\varphi_N) + \mathrm{tr}_e(\varphi_L)$. The proof of the proposition is completed.

In the sequel, S_e will stand for the semi-simple Λ-module $e\Lambda/eJ$. The following observation is essential in our investigation.

1.5. Lemma. *Let e be an idempotent in Λ. If S_e is of finite injective dimension, then the e-trace is defined for every endomorphism in* $\mathrm{mod}\Lambda$.

Proof. Suppose that S_e is of finite injective dimension n. Let M be a module in $\mathrm{mod}\Lambda$ with a minimal projective resolution

$$\mathcal{P}_M: \qquad \cdots \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

For each $i > n$, we have $\mathrm{Hom}_\Lambda(P_i, S_e) = \mathrm{Ext}^i_\Lambda(M, S_e) = 0$, and hence e annihilates the top of P_i. This shows that $\mathcal{P}_M$ is e-bounded. Therefore, $\mathrm{tr}_e(\varphi)$ is defined for every endomorphism φ of M. The proof of the lemma is completed.

Remark. If Λ is of finite global dimension, then we recover Lenzing's trace function by taking e to be the identity of Λ.

Now, we are able to describe the Hochschild homology group $\mathrm{HH}_0(\Lambda_e)$ in case S_e is of finite injective dimension.

1.6. Theorem. *Let Λ be an artin algebra, and let e be an idempotent in Λ. If S_e is of finite injective dimension, then* $\mathrm{HH}_0(\Lambda_e)$ *is radical-trivial.*

Proof. Suppose that S_e is of finite injective dimension. By Lemma 1.5, the e-trace is defined for every endomorphism in $\mathrm{mod}\Lambda$. Let $x \in \Lambda$ be such that $\bar{x} = x + \Lambda(1-e)\Lambda$ lies in the radical of Λ_e, which is $(J + \Lambda(1-e)\Lambda)/\Lambda(1-e)\Lambda$. Hence, $\bar{x} = \bar{a}$ for some $a \in J$. Let $r > 0$ be such that $a^r = 0$, and consider the chain

$$0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = \Lambda,$$

of submodules of Λ, where $M_i = a^i\Lambda$, $i = 0, \ldots, r$. Let $\varphi_0 : \Lambda \to \Lambda$ be the left multiplication by a. Since $\varphi_0(M_i) \subseteq M_{i+1}$, we see that φ_0 induces morphisms $\varphi_i : M_i \to M_i$, $i = 1, \ldots, r$, such that

$$\begin{tikzcd}
0 \arrow[r] & M_{i+1} \arrow[r] \arrow[d, "\varphi_{i+1}"] & M_i \arrow[r] \arrow[d, "\varphi_i"] & M_i/M_{i+1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & M_{i+1} \arrow[r] & M_i \arrow[r] & M_i/M_{i+1} \arrow[r] & 0
\end{tikzcd}$$

commutes. By Proposition 1.4, $\mathrm{tr}_e(\varphi_i) = \mathrm{tr}_e(\varphi_{i+1})$, for $i = 0, 1, \ldots, r-1$. Applying Proposition 1.1(5), we get

$$\bar{a} + [\Lambda_e, \Lambda_e] = H_e(a + [\Lambda, \Lambda]) = H_e(\mathrm{tr}(\varphi_0)) = \mathrm{tr}_e(\varphi_0) = \mathrm{tr}_e(\varphi_r) = 0,$$

that is, $\bar{x} = \bar{a} \in [\Lambda_e, \Lambda_e]$. The proof of the theorem is completed.

Taking e to be the identity of Λ, we recover the following well known result; see, for example, [13].

1.7. Corollary. *If Λ is an artin algebra of finite global dimension, then* $\mathrm{HH}_0(\Lambda)$ *is radical-trivial.*

Let Λ be a finite dimensional algebra over a field of characteristic zero. If Λ is of finite global dimension, then all the Hochschild homology groups $\mathrm{HH}_i(\Lambda)$ with $i \geq 1$ vanish; see [13]. However, in the situation as in Theorem 1.6, even if Λ is of finite global dimension, Λ_e may be of infinite global dimension with non-vanishing higher Hochschild homology groups.

Example. Let $\Lambda = kQ/I$, where k is a field, Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & 2 \arrow[d, "\beta"] \\ 4 \arrow[r, "\delta"] & 3 \arrow[ul, "\varepsilon"'] \end{tikzcd}$$

and I is the ideal in the path algebra kQ generated by $\alpha\beta - \gamma\delta, \beta\varepsilon, \delta\varepsilon, \varepsilon\alpha$. It is easy to see that Λ is of finite global dimension. Let e be the sum of the primitive idempotents in Λ associated to the vertices $1, 2, 3$. By Theorem 1.6, $\mathrm{HH}_0(\Lambda_e)$ is radical-trivial. On the other hand, Λ_e is a Nakayama algebra of infinite global dimension, and a direct computation shows that $\mathrm{HH}_2(\Lambda_e)$ does not vanish; see [11].

2. Main results

The main objective of this section is to apply the previously obtained result to solve the strong no loop conjecture for finite dimensional algebras over an algebraically closed field.

We start with an artin algebra Λ and a primitive idempotent e in Λ. We shall say that Λ is *locally commutative at* e if $e\Lambda e$ is commutative and that Λ is *locally commutative* if it is locally commutative at every primitive idempotent. Moreover, e is called *basic* if $e\Lambda$ is not isomorphic to any direct summand of $(1-e)\Lambda$. In this terminology, Λ is basic if and only if its primitive idempotents are all basic.

2.1. Proposition. *Let Λ be an artin algebra, and let e be a basic primitive idempotent in Λ such that Λ/J^2 is locally commutative at $e + J^2$. If S_e is of finite projective or injective dimension, then* $\mathrm{Ext}^1_\Lambda(S_e, S_e) = 0$.

Proof. Firstly, assume that S_e is of finite injective dimension. For proving that $\mathrm{Ext}^1_\Lambda(S_e, S_e) = 0$, it suffices to show that $eJe/eJ^2e = 0$. Let $a \in eJe$. Then $a + \Lambda(1-e)\Lambda \in [\Lambda_e, \Lambda_e]$ by Theorem 1.6. Since e is basic, $e\Lambda(1-e)\Lambda e \subseteq eJ^2e$. This yields an algebra homomorphism

$$f : \Lambda_e \to e\Lambda e/eJ^2e : x + \Lambda(1-e)\Lambda \mapsto exe + eJ^2e.$$

Thus, $a + eJ^2e = f(a + \Lambda(1-e)\Lambda)$ lies in the commutator group of $e\Lambda e/eJ^2e$. On the other hand, $e\Lambda e/eJ^2e \cong (e + J^2)(\Lambda/J^2)(e + J^2)$, which is commutative. Therefore, $a + eJ^2e = 0$, that is, $a \in eJ^2e$. The result follows in this case.

Next, assume that S_e is of finite projective dimension. Let D be the standard duality between $\mathrm{mod}\Lambda$ and $\mathrm{mod}\Lambda^{\mathrm{op}}$. Then $D(S_e)$ is a simple Λ^{op}-module of finite injective dimension, which is supported by the primitive idempotent e^{o} corresponding to e. Since e^{o} is basic such that the quotient of Λ^{op} modulo the square of its radical is locally commutative at the class of e^{o}, we have $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(D(S_e), D(S_e)) = 0$, and consequently, $\mathrm{Ext}^1_\Lambda(S_e, S_e) = 0$. The proof of the lemma is completed.

Remark. In particular, Proposition 2.1 establishes the strong no loop conjecture for basic artin algebras Λ such that Λ/J^2 is locally commutative.

Now we specialize the preceding result to finite dimensional algebras over a field.

2.2. Theorem. *Let Λ be a finite dimensional algebra over a field k, and let S be a simple Λ-module of k-dimension one. If S is of finite projective or injective dimension, then* $\mathrm{Ext}^1_\Lambda(S,S)=0$.

Proof. Let e be a primitive idempotent in Λ which does not annihilate S. Then Λ admits a complete set $\{e_1,\ldots,e_n\}$ of orthogonal primitive idempotents with $e=e_1$. We may assume that $e_1\Lambda,\ldots,e_r\Lambda$, with $1\le r\le n$, are the non-isomorphic indecomposable projective modules in $\mathrm{mod}\Lambda$. Then

$$\Lambda/J \cong M_{n_1}(D_1)\times\cdots\times M_{n_r}(D_r),$$

where $D_i=\mathrm{End}_\Lambda(e_i\Lambda/e_iJ)$ and n_i is the number of indices j with $1\le j\le n$ such that $e_j\Lambda\cong e_i\Lambda$, for $i=1,\ldots,r$. Observe that S is a simple $M_{n_1}(D_1)$-module, and hence $S\cong D_1^{n_1}$. Since S is one dimensional over k, it is one dimensional over D_1. In particular, $n_1=1$. That is, e is a basic primitive idempotent. Moreover, $e\Lambda e/eJe\cong Se\cong k$. Thus, for $x_1,x_2\in e\Lambda e$, we can write $x_i=\lambda_i e+a_i$, where $\lambda_i\in k$ and $a_i\in eJe$, $i=1,2$. This yields $x_1x_2-x_2x_1=a_1a_2-a_2a_1\in eJ^2e$. Therefore, $e\Lambda e/eJ^2e$ is commutative, and so is $(e+J^2)(\Lambda/J^2)(e+J^2)$. Now the result follows immediately from Proposition 2.1. The proof of the theorem is completed.

Remark. A finite dimension algebra over a field is called *elementary* if its simple modules are all one dimensional over the base field, or equivalently, it is isomorphic to an algebra given by a quiver with relations; see [1]. It is well known that a finite dimensional algebra over an algebraically closed field is Morita equivalent to an elementary algebra. In this connection, Theorem 2.3 establishes the strong no loop conjecture for finite dimensional elementary algebras over any field, and consequently, for finite dimensional algebras over an algebraically closed field.

We shall extend our results further. For this purpose, some more terminology is needed. From now on, let Λ stand for a finite dimensional elementary algebra over a field k, which is isomorphic to an algebra given by a quiver with relations. To simplify the notation, assume that $\Lambda=kQ/I$, where Q is a finite quiver, kQ is the path algebra of Q over k, and I is an admissible ideal in kQ. Recall that I is *admissible* if $(kQ^+)^n\subseteq I\subseteq (kQ^+)^2$ for some $n\ge 2$, where kQ^+ is the ideal in kQ generated by the arrows. Consider $\rho=\lambda_1p_1+\cdots+\lambda_rp_r\in I$, where the p_i are distinct paths in Q from one fixed vertex to another, and the λ_i are non-zero scalars in k. We say that ρ is a *minimal relation* for Λ if $\sum_{i\in\Omega}\lambda_ip_i\notin I$ for any $\Omega\subset\{1,\ldots,r\}$. Observe that a minimal relation in this sense does not necessarily lie in a minimal set of generators of I. For instance, a path p in Q is a minimal relation for Λ if and only if $p\in I$. Moreover, a path p in Q is said to be *non-zero* in Λ if $p\notin I$; and *free* in Λ if p is not a summand of any minimal relation for Λ.

Now, let $\sigma=\alpha_1\alpha_2\cdots\alpha_r$ be an oriented cycle in Q, where the α_i are arrows. We denote by $\mathrm{supp}(\sigma)$ the set of vertices occurring as starting points of $\alpha_1,\ldots,\alpha_r$, and define the *idempotent supporting* σ to be the sum of all primitive idempotents in Λ associated to the vertices in $\mathrm{supp}(\sigma)$. Furthermore, the *cyclic permutations* of σ are the oriented cycles $\sigma_1=\sigma$, $\sigma_2=\alpha_2\cdots\alpha_r\alpha_1$, …, and $\sigma_r=\alpha_r\alpha_1\cdots\alpha_{r-1}$. Now, we say that σ is *cyclically non-zero* (respectively, *cyclically free*) *in* Λ if each of $\sigma_1,\ldots,\sigma_r$ is non-zero (respectively, free) in Λ. For example, a loop in Q is always cyclically free in Λ.

2.3. Theorem. *Let $\Lambda = kQ/I$ with Q a finite quiver and I an admissible ideal in kQ, and let σ be an oriented cycle in Q with supporting idempotent e in Λ. If σ is cyclically free in Λ, then S_e is of infinite projective and injective dimensions.*

Proof. Suppose that σ is cyclically free in Λ. If σ is a power of a shorter oriented cycle δ, then it is easy to see that δ is also cyclically free in Λ and $\mathrm{supp}(\delta) = \mathrm{supp}(\sigma)$. Hence, we may assume that σ is not a power of any shorter oriented cycle. It is then well known that the cyclic permutations $\sigma_1, \ldots, \sigma_r$ of σ, where $\sigma_1 = \sigma$, are pairwise distinct.

For any $x \in kQ$, denote by $\bar{x}$ its class in Λ and by $\tilde{x}$ the class of $\bar{x}$ in Λ_e. Let W be the vector subspace of Λ_e spanned by the classes $\tilde{p}$, where p ranges over the paths in Q different from $\sigma_1, \ldots, \sigma_r$. Then, there exist paths $p_1, \ldots, p_m$ in Q different from $\sigma_1, \ldots, \sigma_r$ such that $\{\tilde{p}_1, \ldots, \tilde{p}_m\}$ is a k-basis of W. We claim that $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r, \tilde{p}_1, \ldots, \tilde{p}_m\}$ is a k-basis of Λ_e. Indeed, it clearly spans Λ_e. Assume that

$$\textstyle\sum_{i=1}^{r} \lambda_i \tilde{\sigma}_i + \sum_{j=1}^{m} \nu_j \tilde{p}_j = \tilde{0}, \ \lambda_i, \nu_j \in k.$$

That is, $\sum_{i=1}^{r} \lambda_i \bar{\sigma}_i + \sum_{j=1}^{m} \nu_j \bar{p}_j \in \Lambda(1-e)\Lambda$. Then

$$\textstyle\sum_{i=1}^{r} \lambda_i \, \bar{\sigma}_i + \sum_{j=1}^{m} \nu_j \, \bar{p}_j = \sum_{l=1}^{s} \mu_l \, \bar{q}_l, \ \ \mu_l \in k,$$

where $q_1, \ldots, q_s$ are some distinct paths in Q passing through a vertex not in $\mathrm{supp}(\sigma)$. Fix some t with $1 \le t \le r$. Letting ε_t be the trivial path in Q associated to the starting point a_t of σ_t, we get

$$\rho = \textstyle\sum_{i=1}^{r} \lambda_i (\varepsilon_t \, \sigma_i \, \varepsilon_t) + \sum_{j=1}^{m} \nu_j (\varepsilon_t \, p_j \, \varepsilon_t) - \sum_{l=1}^{s} \mu_l (\varepsilon_t \, q_l \, \varepsilon_t) \in I.$$

Note that the non-zero elements of the $\varepsilon_t \sigma_i \varepsilon_t$, $\varepsilon_t p_j \varepsilon_t$, $\varepsilon_t \, q_l \, \varepsilon_t \in kQ$ are distinct oriented cycles from a_t to a_t. If $\lambda_t \neq 0$, then $\lambda_t(\varepsilon_t \sigma_t \varepsilon_t)$, that is $\lambda_t \sigma_t$, would be a summand of a minimal non-zero summand ρ' of ρ with $\rho' \in I$. By definition, ρ' is a minimal relation for Λ, contrary to the hypothesis that σ is cyclically free in Λ. Therefore, $\lambda_t = 0$. This shows that $\lambda_1, \ldots, \lambda_r$ are all zero, and consequently, so are $\nu_1, \ldots, \nu_m$. Our claim is established. Suppose now that $\tilde{\sigma} \in [\Lambda_e, \Lambda_e]$. Then

$$\tilde{\sigma} = \textstyle\sum_{i=1}^{n} \eta_i (\tilde{u}_i \tilde{v}_i - \tilde{v}_i \tilde{u}_i), \tag{1}$$

where $\eta_i \in k$ and $u_i, v_i \in \{\sigma_1, \ldots, \sigma_r, p_1, \ldots, p_m\}$. For each $1 \le i \le n$, we see easily that $u_i v_i \notin \{\sigma_1, \ldots, \sigma_r\}$ if and only if $v_i u_i \notin \{\sigma_1, \ldots, \sigma_r\}$, and in this case, $\tilde{u}_i \tilde{v}_i - \tilde{v}_i \tilde{u}_i \in W$. Therefore, Equation (1) becomes

$$\tilde{\sigma} = \textstyle\sum \eta_{ij} (\tilde{\sigma}_i - \tilde{\sigma}_j) + w, \tag{2}$$

where $\eta_{ij} \in k$ and $w \in W$. Let L be the linear form on Λ_e, which sends each of $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r$ to 1 and vanishes on W. Since $\sigma = \sigma_1$, applying L to Equation (2) yields $1 = 0$, a contradiction. Therefore, the class of $\tilde{\sigma}$ in $\mathrm{HH}_0(\Lambda_e)$ is non-zero. Since $\tilde{\sigma}$ lies in the radical of Λ_e, by Theorem 1.6, S_e is of infinite projective and injective dimensions. The proof of the theorem is completed.

Example. Let $\Lambda = kQ/I$, where Q is the following quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, "\varepsilon"'] & 2 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\gamma"] & 3 \arrow[l, bend left, "\delta"] \arrow[r, bend left, "\mu"] & 4 \arrow[l, bend left, "\nu"] \end{tikzcd}$$

and I is the ideal in kQ generated by $\alpha\beta, \delta\gamma, \beta\varepsilon, \varepsilon\beta, \nu\delta, \nu\mu, \mu\nu, \gamma\mu$, $\alpha\gamma\delta\beta\alpha\gamma - \varepsilon\gamma$. It is easy to see that the oriented cycle $\beta\alpha\gamma\delta$ is cyclically free in Λ. By Theorem 2.3, one of the simple modules S_1, S_2, S_3 is of infinite projective dimension, and one is of infinite injective dimension.

2.4. Corollary. *Let $\Lambda = kQ/I$, where Q is a finite quiver and I is an admissible ideal in kQ. If Q contains an oriented cycle which is cyclically free in Λ, then Λ is of infinite global dimension.*

An admissible ideal I in kQ is called *monomial* if it is generated by some paths. In this case, every minimal relation for Λ is a multiple of a path in Q. Therefore, an oriented cycle in Q is cyclically free in Λ if and only if it is cyclically non-zero in Λ. This yields the following consequence, which can also be derived from some results stated in [11].

2.5. Corollary. *Let $\Lambda = kQ/I$, where Q is a finite quiver and I is a monomial ideal in kQ. If Q contains an oriented cycle which is cyclically non-zero in Λ, then Λ is of infinite global dimension.*

Example. Consider $\Lambda = kQ/I$, where Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and I is the ideal in kQ generated by $\alpha\beta$. It is easy to see that Λ is of global dimension two. Observe that Q contains an oriented cycle $\beta\alpha$ which is non-zero but not cyclically non-zero in Λ.

To conclude, we would like to draw the reader's attention to the following even stronger version of the strong no loop conjecture.

2.6. Extension Conjecture. *Let S be a simple module over an artin algebra. If $\mathrm{Ext}^1(S,S)$ is non-zero, then $\mathrm{Ext}^i(S,S)$ is non-zero for infinitely many integers i.*

This conjecture was originally posed in [14] under the name of *extreme no loop conjecture*. It remains open except for monomial algebras and special biserial algebras; see [6, 14].

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