Shapes of Connected Components of the Auslansder-Reiten Quivers of Artin Algebras

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To the memory of Maurice Auslander

Introduction

The aim of these notes is to report some new developments on the problem of describing all possible shapes of the connected components of the Auslander-Reiten quiver Γ_A of an artin algebra A. The problem is interesting since the shapes of these components carry some important information of the module category of A. For instance the algebra A is hereditary if and only if Γ_A has a connected component of shape $\mathbf{N}\Delta$ where Δ is a quiver without oriented cycles such that the number of its vertices is the same as that of simple A-modules. More importantly, by analyzing the structure of Auslander-Reiten components, Riedtmann classified the self-injective algebras of finite representation type [**44, 45, 46**], and Erdmann did the same for the blocks of finite groups with a dihedral or semidihedral defect group. And remarkably Erdmann has recently showed that the representation type of a block of a finite group is determined by the shapes of the connected components of its Auslander-Reiten quiver [**26**]. More generally in any preprojective or preinjective Auslander-Reiten components, modules are determined by their composition factors and the maps are sums of composites of irreducible maps [**29**]. Furthermore modules in a quasi-serial Auslander-Reiten component behave like serial modules [**47**].

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For some classes of algebras (namely, hereditary algebras [**2, 23, 47**], tilted algebras [**28, 35, 38, 51**], tubular algebras [**49**] and group algebras [**14, 26, 43, 56**]), the shapes of the connected components of their Auslander-Reiten quivers have been described completely. For a general artin algebra A one approach to the problem, which was initialized by Riedtmann [**44**], is to delete the DTr-orbits containing projective or injective modules from Γ_A to obtain a well-behaved subquiver ${}_s\Gamma_A$, called the *stable part* of Γ_A, and then to recover Γ_A from ${}_s\Gamma_A$. The possible shapes of the connected components of ${}_s\Gamma_A$ are described by the works of Riedtmann [**44**], Todorov [**55**], Happel-Preiser-Ringel [**30**] and Zhang [**58**].

The disadvantage of investigating ${}_s\Gamma_A$ is that the stable part of a connected component of Γ_A does not contain the most important modules (that is, projective or injective modules), and sometimes it is even empty. Thus we use replacements for ${}_s\Gamma_A$, which are almost as well-behaved, but carry more information of Γ_A. We delete from Γ_A the DTr-orbits of projective modules to obtain the *left stable part* ${}_l\Gamma_A$ of Γ_A and delete the TrD-orbits of injective modules to get the *right stable part* ${}_r\Gamma_A$. In these notes we shall present a complete description of the possible shapes of the connected components of the quivers ${}_l\Gamma_A$, ${}_r\Gamma_A$ and some applications.

Most of the results in these notes are reformulations of those found in [**36, 37, 38, 39, 40**] with shorter proofs. However, there exist also some new results, namely Proposition 3.1 and Theorem 5.6. We are indebted to Skowronski for some useful discussions.

1. Degrees of Irreducible Maps

Throughout these notes, we denote by A a fixed artin algebra and by $\Re$ the Jacobson radical of $\operatorname{mod} A$, the category of finitely generated right A-modules. Recall that for $m > 0$, the m-th power $\Re^m$ of $\Re$ is defined so that for any modules X, Y in $\operatorname{mod} A$, $\Re^m(X, Y)$ consists of the maps $X \to Y$ which can be written as a sum of composites of m maps in $\Re$, and the infinite radical of $\operatorname{mod} A$ is defined to be the intersection of the $\Re^m$ with $m > 0$.

We denote by Γ_A the Auslander-Reiten quiver of A and by τ and τ^- the Auslander-Reiten translations DTr and TrD respectively. We do not distinguish between an indecomposable module X in $\operatorname{mod} A$ and the corresponding vertex $[X]$, that is the isoclass of X in Γ_A. We shall use freely the standard notions and results of Auslander-Reiten theory which can be found in [**3, 4, 5**].

We devote this section to introduce the notion of degrees of an irreducible map and study some of their properties. This notion emerged from a discussion with Brenner and Butler on the problem as to when the composite of n irreducible maps falls into $\Re^{n+1}$. A partial solution to this problem is the following result of Igusa and Todorov.

1.1. Proposition [**33**]. *Let A be an artin algebra, and let*

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$$

be a chain of irreducible maps between indecomposable modules in $\operatorname{mod} A$. *If* $X_{i-1} \ncong \operatorname{DTr} X_{i+1}$ *for all* $0 < i < n$, *then* $f_1 f_2 \cdots f_n$ *is not in* $\Re^{n+1}$.

On the other hand we provide an example suggested by Skowronski where the composite of two irreducible maps is a non-zero map in the infinite radical. Let K be a field, and let B be the K-algebra given by the bound quiver consisting of one vertex with two loops x, y which satisfy the relations $x^2 = y^2 = xy = yx = 0$. Let $\lambda \in K^*$, and let M be the 2-dimensional representation with

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

Note that M is the quasi-simple module of a homogeneous tube, and the endomorphism

$$\eta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of M factors through the simple representation. So η is in the infinite radical. Let

$$0 \to M \xrightarrow{f} E \xrightarrow{g} M \to 0$$

be an almost split sequence. Then $\eta = \phi g$ for some $\phi : M \to E$. If ϕ is not irreducible, then $f + \phi$ is irreducible and $\eta = (f + \phi) g$.

1.2. Definition. Let $f : X \to Y$ be an irreducible map in $\operatorname{mod} A$. Define the *left degree* $d_l(f)$ of f to be infinity if for any integer $n \geq 1$ and any map

$\theta : M \to X$ in $\Re^n \backslash \Re^{n+1}$, we have $\theta f \notin \Re^{n+2}$. Otherwise it is defined to be the least positive integer m such that there exists some $\theta \in \Re^m \backslash \Re^{m+1}$ with $\theta f \in \Re^{m+2}$. We define the *right degree* $d_r(f)$ of f in a dual manner.

For example, if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an almost split sequence, then $d_l(g) = 1$ and $d_r(f) = 1$.

The following lemma is an immediate consequence of the above definition.

1.3. Lemma. *The following statements hold for an artin algebra A:*

(1) *Let $f : X \to Y$ be an irreducible map in* mod A. *If Y' is a direct summand of Y and g is the co-restriction of f to Y', then $d_l(g) \le d_l(f)$. Dually if X' is a direct summand of X and h is the restriction of f to X', then $d_r(h) \le d_r(f)$.*

(2) *Each chain of irreducible maps in* mod A *of length n with the composite in $\Re^{n+1}$ contains at least one maps of finite left degree and one of finite right degree.*

The following lemma and its dual are crucial in the study of degrees of irreducible maps.

1.4. Lemma. *Let A be an artin algebra, and let $\theta : M \to X$ be a map in $\Re^n \backslash \Re^{n+1}$ with $n \ge 1$ an integer. Suppose that $f : X \to Y$ is an irreducible map in* mod A *with Y indecomposable. If $\theta f \in \Re^{n+2}$, then*

(1) *Y is not projective, and*

(2) *for an almost split sequence $0 \to \mathrm{DTr}\, Y \xrightarrow{(g,g')} X \oplus X' \xrightarrow{\binom{f}{f'}} Y \to 0$ in* mod A, *there exists a map $\zeta : M \to \mathrm{DTr}\, Y \notin \Re^n$ such that $\theta + \zeta g \in \Re^{n+1}$ and $\zeta g' \in \Re^{n+1}$.*

Proof. Assume that $\theta f \in \Re^{n+2}$. Then $\theta f = st$ with $s \in \Re^{n+1}$ and $t \in \Re$. Let

$$\begin{pmatrix} f \\ f' \end{pmatrix} : X \oplus X' \to Y$$

be a sink map for Y. Then t has a factorization $t = (u, u')\binom{f}{f'}$. Hence

$$(su - \theta, su')\begin{pmatrix} f \\ f' \end{pmatrix} = 0.$$

Since $su - \theta \neq 0$, Y is not projective. Let

$$0 \to \tau Y \xrightarrow{(g,g')} X \oplus X' \xrightarrow{\binom{f}{f'}} Y \to 0$$

be an almost split sequence. Then there exists a map $\zeta : M \to \tau Y$ such that $(su - \theta, su') = \zeta(g, g')$. Hence $(\theta + \zeta g, \zeta g) = (su, su') \in \Re^{n+1}$. Moreover $\theta \notin \Re^{n+1}$ implies that $\zeta \notin \Re^n$. The proof is completed.

Let $X_0 \to X_1 \to \cdots \to X_n$ be a path in Γ_A. Recall that the path is *sectional* if $X_{i-1} \neq DTr\ X_{i+1}$ for all $0 < i < n$, and more generally it is *pre-sectional* if for all $0 < i < n$, either X_{i+1} is projective or otherwise $DTr\ X_{i+1} \oplus X_{i-1}$ is a direct summand of the domain of a sink map for X_i.

As an immediate consequence of Lemma 1.4, we have the following.

1.5. Corollary. *Let A be an artin algebra, and let $f : X \to Y$ be an irreducible map in* mod A *of finite left degree. Assume that Y is indecomposable and that*

$$Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y$$

is a pre-sectional path in Γ_A such that $X \oplus Y_1$ is a direct summand of the domain of a sink map for Y. Then the Y_i are not projective, and for each $1 \le i \le n$, there exists an irreducible map f_i : $\mathrm{DTr}\ Y_{i-1} \to Y_i$ such that $d_l(f_n) < \cdots < d_l(f_1) < d_l(f)$. In particular $d_l(f) > n$.

Recall that the valuation (d_{XY}, d'_{XY}) of an arrow $X \to Y$ in Γ_A is defined so that d_{XY} is the multiplicity of Y in the codomain of a source map for X that is the dimension of $\Re(X, Y)/\Re^2(X, Y)$ over $\mathrm{End}(Y)/\Re(\mathrm{End}(Y))$, and d'_{XY} is the multiplicity of X in the domain of a sink map for Y that is the dimension of $\Re(X, Y)/\Re^2(X, Y)$ over $\mathrm{End}(X)/\Re(\mathrm{End}(X))$.

1.6. Proposition. *Let A be an artin algebra, and let $X \to Y$ be an arrow in Γ_A with valuation $(d_{XY},\ d'_{XY})$. If $d_{XY} > 1$ and $d'_{XY} > 1$, then all irreducible maps $f : X \to Y$ have infinite left and right degrees.*

Proof. Assume that $d_{XY}, d'_{XY} > 1$ and $f : X \to Y$ is an irreducible map. For $n \ge 0$, the path

$$\tau^n X \to \tau^n Y \to \tau^{n-1} X \to \cdots \to \tau Y \to X \to Y$$

is pre-sectional. Note that $X \oplus X$ is a direct summand of the domain of a sink map for Y. If $\tau^n X$ or $\tau^n Y$ is projective for some n, then f has infinite left degree by Corollary 1.5. Otherwise $d_l(f) > n$ for all n. Hence the left degree of f is infinite. Finally f has infinite right degree by the dual of Corollary 1.5.

1.7. Corollary. *Let A be an artin algebra, and let $f : X \to Y$ and $g : X \to Y$ be irreducible maps in* mod A *with X, Y indecomposable. Then $d_l(f) = d_l(g)$ and $d_r(f) = d_r(g)$.*

Proof. Let (d_{XY}, d'_{XY}) be the valuation of the arrow $X \to Y$ in Γ_A. If $d_{XY} > 1$ and $d'_{XY} > 1$ then by Proposition 1.6, $d_l(f), d_l(g), d_r(f)$ and $d_r(g)$ are all infinite. Otherwise $f - ga \in \Re^2$ for some $a \in$Aut(Y) or $f - bg \in \Re^2$ for some $b \in$Aut(X). It is now clear that $d_l(f) = d_l(g)$, $d_r(f) = d_r(g)$.

By the above corollary the following definition makes sense.

1.8. Definition. Let $X \to Y$ be an arrow in Γ_A. Define the *left degree* and the *right degree* of the arrow $X \to Y$ to be those of an irreducible map $f : X \to Y$.

1.9. Proposition. *Let A be an artin algebra. Then each oriented cycle in Γ_A contains at least one arrow of finite left degree and at least one of finite right degree.*

Proof. Let $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n = X_0$ be a cycle of irreducible maps between indecomposable modules in mod A. Then $(f_1 \cdots f_n)^r = 0$ for some $r > 0$. Hence at least one of the f_i is of finite left degree and one is of finite right degree. This establishes the proposition.

1.10. Proposition. *Let A be an artin algebra, and let*

$$f = (f_1,\ f_2) : X \to Y_1 \oplus Y_2$$

be an irreducible map in mod A *with X, Y_1 and Y_2 all indecomposable. If f has finite left degree, then there exists an irreducible map*

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : \mathrm{DTr}\, Y_1 \oplus \mathrm{DTr}\, Y_2 \to X$$

with $d_l(g) < d_l(f)$.

Proof. Let $d_l(f) = m$. Then there exists $\theta : M \to X \in \Re^m \backslash \Re^{m+1}$ such that $\theta f \in \Re^{m+2}$. So $\theta f_i \in \Re^{m+2}$ for $i = 1, 2$. Let

$$0 \to \tau Y_i \xrightarrow{(g_i, p_i)} X \oplus Z_i \xrightarrow{\binom{f_i}{q_i}} Y_i \to 0$$

be an almost split sequence for $1 \leq i \leq 2$. By Lemma 1.4, there exists $\zeta_i : M \to \tau Y_i \notin \Re^m$ such that $\theta + \zeta_i g_i \in \Re^{m+1}$ for $1 \leq i \leq 2$. Hence we have

$$(\zeta_1, -\zeta_2) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \Re^{m+1}.$$

Since $(\zeta_1, -\zeta_2) \notin \Re^m$, it suffices to show that

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : \tau Y_1 \oplus \tau Y_2 \to X$$

is irreducible. Assume that this was not the case. Then we may assume that $Y_1 = Y_2$ and $Z_1 = Z_2$. Furthermore $g_1 = g_2 a - \eta$ where $a \in \mathrm{Aut}(X)$ and $\eta \in \Re^2$. We now have factorizations $\eta = g_1 u_1 + p_1 u_2$ with $u_1, u_2 \in \Re$ and $a p_2 = g_1 v_1 + p_1 v_2$. Note $v_1 \in \Re$ since X is not a direct summand of Z_1 by Proposition 1.6. So

$$a(g_2, p_2) = (g_1, p_1) \begin{pmatrix} 1 + u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

and hence $\left(\begin{smallmatrix} 1+u_1 & v_1 \\ u_2 & v_2 \end{smallmatrix}\right)$ is an automophism. Thus there exists $b \in \mathrm{Aut}(Z)$ such that

$$\begin{pmatrix} 1 + u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} f_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ q_1 \end{pmatrix} b.$$

Then $f_1 b - f_2 = u_1 f_2 + v_1 q_2 \in \Re^2$, and hence $(f_1, f_2) : X \to Y_1 \oplus Y_2$ is not irreducible, which is a contradiction.

1.11. Definition. Let $X \to Y$ be an arrow in $\varGamma_A$. The *global left degree* of $X \to Y$ is the minimum of left degrees of all possible arrows $DTr^n X \to DTr^n Y$, $DTr^{n+1} Y \to DTr^n X$ with $n \geq 0$. The *global right degree* of $X \to Y$ is defined in a dual manner.

A module $X \in \varGamma_A$ is said to be *left stable* if $DTr^n X \neq 0$ for all $n > 0$, *right stable* if $TrD^n X \neq 0$ for all $n > 0$ and *stable* if it is both left and right stable.

1.12. Lemma. *Let A be an artin algebra.*

(1) *Let*

$$\cdots \to X_{i+1} \to X_i \to \cdots \to X_1 \to X_0$$

and

$$\cdots \to Y_{i+1} \to Y_i \to \cdots \to Y_1 \to Y_0$$

be infinite pre-sectional paths in Γ_A containing only left stable modules. If $X_0 = Y_0$ and there exists an irreducible map from $X_1 \oplus Y_1$ to X_0, then the arrow $X_{i+1} \to X_i$ has infinite global left degree for all $i \geq 0$.

(2) *Let*

$$\cdots \to X_i \to \cdots \to X_1 \to X_0 \to X_{-1} \to \cdots \to X_{-i} \to \cdots$$

be a double infinite pre-sectional path in Γ_A containing only left stable modules. Then $X_i \to X_{i-1}$ has infinite global left degree for all integers i.

Proof. (1) follows from Corollary 1.5, and (2) follows from (1).

2. Semi-stable Translation Quivers

In this section we study some general translation quivers. Note that all of our translation quivers are locally finite and admit no multiple arrow; moreover each non-projective vertex has at least one direct predecessor.

2.1. Definition. Let Γ be a connected translation quiver with translation ρ. A connected full subquiver Δ of Γ is said to be a *section* if

S1. There exists no oriented cycle in Δ.

S2. Each ρ-orbit in Γ meets Δ exactly once.

S3. Each path in Γ with end-points in Δ lies completely in Δ.

Remark. (1) If Δ is a section in Γ and i is an integer such that $\rho^i x$ is defined for all $x \in \Delta$, then the full subquiver $\rho^i \Delta$ generated by the vertices $\rho^i x$ with $x \in \Delta$, is also a section in Γ.

(2) In [**17**], Bongartz also defined the notion of a section of a translation quiver. It turns out that the only difference is that in his definition a section can contain periodic vertices.

The following is an immediate consequence of our definition.

2.2. Lemma. *Let Γ be a translation quiver with translation ρ, and let Δ be a section in Γ. If $x \to y$ is an arrow in Γ, then $x \in \Delta$ implies $y \in \Delta$ or $\rho y \in \Delta$, and $y \in \Delta$ implies $x \in \Delta$ or $\rho^- x \in \Delta$.*

Let Δ be a quiver, we denote by Δ_0 the set of vertices and by Δ_1 the set of arrows. Assume that Δ has no oriented cycle. Recall that $\mathbf{Z}\Delta$ is a translation quiver defined as follows: the vertices are the pairs (n, x) with $n \in \mathbf{Z}, x \in \Delta_0$; the arrows are $(n, x) \to (n, y)$ and $(n+1, y) \to (n, x)$ where $n \in \mathbf{Z},\ x \to y \in \Delta$; and the translate of (n, x) is $(n-1, x)$. We denote by $\mathbf{N}\Delta$ the full sub-translation-quiver of $\mathbf{Z}\Delta$ generated by the vertices (n, x) with $n \in \mathbf{N}$ and $x \in \Delta_0$.

It is easy to see from the construction that each copy of Δ in $\mathbf{Z}\Delta$, that is the full subquiver generated by the vertices (n, x) with $x \in \Delta_0$ and n some fixed integer, is a section in $\mathbf{Z}\Delta$. Conversely we have the following observation.

2.3. Proposition. *Let Γ be a translation quiver with translation ρ, and let Δ be a section in Γ. Then Γ is isomorphic to the full subquiver of $\mathbf{Z}\Delta$ generated by the vertices (n, u) with $n \in \mathbf{Z}$ and $u \in \Delta_0$ such that $\rho^n u$ is defined. In particular Γ contains no oriented cycle.*

Proof. By definition Δ contains no ρ-periodic vertex. Let Σ be the subquiver of Γ consisting of all possible arrows $\rho^n u \to \rho^n v$, $\rho^{n+1} v \to \rho^n u$ with $u \to v \in \Delta_1,\ n \in \mathbf{Z}$. Then Σ is isomorphic to the full subquiver of $\mathbf{Z}\Delta$ generated by the vertices (n, u), where $n \in \mathbf{Z}$ and $u \in \Delta_0$ such that $\rho^n x$ is defined. It now suffices to show that $\Gamma_1 = \Sigma_1$.

Let $x \to y \in \Gamma_1$. Then $x = \rho^n u, y = \rho^m v$ with $u, v \in \Delta; m, n \in \mathbf{Z}$. If $m = 0$ or $n = 0$, then $x \to y \in \Sigma$ by Lemma 2.2. Now assume that $m > 0, n > 0$. Then either $u \to \rho^{m-n} v \in \Gamma_1$ or $\rho^{n-m} u \to v \in \Gamma_1$. By S2 and Lemma 2.2, one of the following identities

$$\rho^{m-n} v = v,\ \rho^{m-n+1} v = v,\ \rho^{n-m} u = u,\ \rho^{n-m-1} u = u$$

holds. Thus either $m = n$ or $n = m + 1$. Hence $x \to y \in \Sigma_1$. Similarly we can show that if $m < 0$ and $n < 0$, then $x \to y \in \Sigma_1$. Now assume that $m < 0$ and $n > 0$. Then Γ contains a path

$$v \to \cdots \to \rho^m v = y \to \rho^- x = \rho^{n-1} u \to \cdots \to u,$$

which is a contradiction to S3 and S2. Similarly we can show that the case where $m > 0$ and $n < 0$ can not happen. The proof is completed.

Let Γ be a translation quiver with translation ρ. A vertex $x \in \Gamma$ is said to be *left stable* if $\rho^n x$ is defined for all $n > 0$, and *right stable* if $\rho^n x$ is defined for all $n < 0$, and finally *stable* if $\rho^n x$ is defined for all $n \in \mathbf{Z}$. We say that Γ is *left stable* (*right stable, stable* respectively) if so are all the vertices in Γ.

2.4. Theorem. *Let Γ be a connected left stable translation quiver with translation ρ. Assume that Γ contains at most finitely many injective vertices and no oriented cycle.*

(1) *There exist vertices in Γ which have no injective predecessor.*

(2) *Let $x \in \Gamma$ be a vertex admitting no injective predecessor. Then each ρ-orbit in Γ contains a vertex u such that u is a predecessor of x while $\rho^- u$ is not, and the full subquiver of Γ generated by all such vertices u is a section in Γ with x as a unique sink.*

Proof. Since Γ is connected and left stable, for any vertices v, v' in Γ, there exists $r \geq 0$ such that $\rho^r v$ is a predecessor of v' in Γ.

(1) Pick $w \in \Gamma$. For any injective vertex p in Γ, $\rho^r w$ is a predecessor of p for some $r \geq 0$. Thus p is not a predecessor of $\rho^r w$. Since Γ has at most finitely many injective vertices, there exists $s \geq 0$ such that $\rho^s w$ has no injective predecessor in Γ.

(2) Assume that x is a vertex in Γ admitting no injective predecessor. Let $\mathcal{O}$ be a ρ-orbit in Γ and $y \in \mathcal{O}$. Then $\rho^n y$ is a predecessor of x in Γ for some $n \geq 0$. We now claim that $\mathcal{O}$ contains a vertex which is not a predecessor of x. Assume that this is not the case. Then y is right stable since x has no injective predecessor in Γ, and for each $m \geq 0$, there exists a path

$$\sigma_m : \rho^{-m} y = z_0^m \to z_1^m \to \cdots \to z_{i_m - 1}^m \to z_{i_m}^m = x$$

in Γ. If σ_{m_0} contains only right stable vertices for some $m_0 \geq 0$, then there exists $m_1 > m_0$ such that Γ contains a path from x to $\rho^{-m_1} y$, which contradicts that Γ has no oriented cycle. Hence each σ_m contains a vertex which is not right stable. Thus for each $m \geq 0$, there exists j_m, $0 < j_m \leq i_m$ such that $z_{j_m}^m$ is not right stable but z_k^m is right stable for all k, $0 \leq k < j_m$. Write $z_{j_m}^m = \rho^{r_m} q_m$ with q_m an injective vertex and $r_m \geq 0$. Since Γ contains at most finitely many injective vertices, there exists an injective vertex q such that $z_{j_m}^m = \rho^{r_m} q$ for infinitely many $m \geq 0$. Note that there exists some n_o such that $\rho^{n_0} q$ is a predecessor of y in Γ. Thus for each $r \geq n_0$, Γ has no path from y to $\rho^r q$. So if $z_{j_m}^m = \rho^{r_m} q$, then $0 \leq r_m < n_0$. It follows that there exists s, $0 \leq s < n_0$ such that $z_{j_m}^m = \rho^s q$ for infinitely many $m \geq 0$. That is, there exist infinitely many $m \geq 0$ such that Γ contains a path

$$\delta_m : \rho^{-m} y = z_0^m \to z_1^m \to \cdots \to z_{j_m-1}^m \to z_{j_m}^m = \rho^s q$$

with z_k^m stable for all $0 \leq k < j_m$. Let m' be an integer such that $\delta_{m'}$ exists. Then Γ contains a path from $\rho^s q$ to $\rho^{-m'-j_{m'}} x$. Let $m'' > m' + j_{m'}$ such that $\delta_{m''}$ exists. This gives rise to an oriented cycle in Γ, which is a contradiction. Thus our claim holds.

Let $\{\mathcal{O}_i \,|\, i \in I\}$ be the set of all ρ-orbits in Γ. Then each $\mathcal{O}_i$ contains a unique vertex u_i such that u_i is a predecessor of x but $\rho^- u_i$ is not. Let Δ be the full subquiver generated by the u_i. Then Δ has x as a unique sink and satisfies the properties S1 and S2. Let

$$x_0 \to x_1 \to \cdots \to x_{t-1} \to x_t$$

be a path in Γ with $x_0, x_t \in \Delta$. Then the x_i are all predecessors of x since $x_t \in \Delta$, and hence are all non-injective. Therefore if $\rho^- x_i$ is a predecessor of x for some $i < t$, then so is $\rho^- x_0$, which contradicts that $x_0 \in \Delta$. That is, the x_i are all in Δ. So Δ is a section in Γ.

We shall now consider semi-stable translation quivers with oriented cycles. We first recall the constructions of coray insertion and ray insertion from [**22**].

A vertex x in a translation quiver Γ is called a *coray vertex* if there exists an infinite sectional path

$$\cdots \to x_n \to x_{n-1} \to \cdots \to x_2 \to x_1 = x,$$

called a *coray*, in Γ with pairwise different vertices such that for each integer $i > 0$, the path $x_{i+1} \to x_i \to \cdots \to x_2 \to x_1$ is the only sectional path of length i in Γ which ends with x_1.

Let Γ be a translation quiver with translation ρ, and let x be a *coray vertex* in Γ with a coray as above. For a positive integer n, we construct a new translation quiver $\Gamma[x, n]$ by inserting n corays into Γ as follows. The vertices of $\Gamma[x, n]$ are those of Γ together with the pairs $(i,\ j)$ with $i \geq 1$ and $1 \leq j \leq n$. The arrows of $\Gamma[x, n]$ are those of Γ, excluding the arrows $y \to x_i$ with $i \geq 1$ other than $x_{i+1} \to x_i$, together with the following arrows:
$(i+1,\ j) \to (i,\ j)$ for $i \geq 1$ and $1 \leq j \leq n$;
$(i,\ j+1) \to (i+1,\ j)$ for $i \geq 1$ and $1 \leq j < n$;

$(n+i-1,\ 1) \to x_i$ for $i \geq 1$
and $y \to (i,\ n)$ whenever $y \to x_i$ is an arrow in Γ other than $x_{i+1} \to x_i$. The translation ρ' of $\Gamma[x,n]$ is defined so that for a vertex $z \in \Gamma$, $\rho' z = \rho z$ if z is different from all x_i such that ρz is defined, and $\rho' z = (n+i,\ 1)$ if $z = x_i$ for some $i \geq 1$, and $\rho'(i,\ j) = (i,\ j+1)$ for $i \geq 1$ and $1 \leq j < n$, and finally $\rho'(i,\ n) = \rho x_i$ if ρx_i is defined. We call $\Gamma[x,n]$ a translation quiver obtained from Γ by *coray insertion*. From the construction we see that if Γ is left stable, then $\Gamma[x,n]$ is also left stable and contains n new projective vertices $(1,1), \ldots, (n,1)$.

A *ray vertex* of a translation quiver is the dual concept to a coray vertex. Let Γ be a translation quiver with a ray vertex x. For a positive integer n, we define a new translation quiver $[x,n]\Gamma$ by a construction dual to that of $\Gamma[x,n]$, and call $[x,n]\Gamma$ a translation quiver obtained from Γ by *ray insertion*. And if Γ is right stable, then $[x,n]\Gamma$ is right stable and contains n new injective vertices.

Recall that a translation quiver is called a *stable tube* if it is isomorphic to $\mathbf{Z}\mathrm{A}_\infty/(\rho^n)$ for some $n > 0$, where ρ is the translation of $\mathbf{Z}\mathrm{A}_\infty$.

2.5. Definition. A translation quiver Γ is called a *coray tube* if it is obtained from a stable tube by a sequence of coray insertions, and a *ray tube* if it is obtained from a stable tube by a sequence of ray insertions.

Let Γ be a connected left stable translation quiver with translation ρ. It is not difficult to check that Γ is a coray tube if and only if each vertex in Γ has at most two direct predecessors and there exists a sectional path

$$\rho^r x_1 \to x_s \to \cdots \to x_2 \to x_1$$

in Γ, where $r > s \geq 1$ and $x_i = \rho^{n_i} q_i$ for $1 \leq i \leq s$ with the q_i distinct injective vertices and $n_1 \leq n_2 \leq \cdots \leq n_s$. In this case Γ is obtained by inserting s corays into the stable tube of rank $r-s$ in a sequence of t coray insertions, where t is the number of distinct integers in $\{n_1, n_2, \ldots, n_r\}$.

3. Semi-stable Components

In this section we apply the results obtained in the preceding section to study the Auslander-Reiten quiver Γ_A.

Recall that the *stable part* ${}_s\Gamma_A$ of Γ_A is the full subquiver generated by the stable modules, and the connected components of the quiver ${}_s\Gamma_A$ are called the *stable components* of Γ_A. Accordingly we define the *left stable part* ${}_l\Gamma_A$ of Γ_A to be the full subquiver generated by the left stable modules, and the *right stable part* ${}_r\Gamma_A$ to be the one generated by the right stable modules. The connected components of the quiver ${}_l\Gamma_A$ are called the *left stable components* of Γ_A, and those of the quiver ${}_r\Gamma_A$ are called the *right stable components*. We refer a *semi-stable component* of Γ_A to a left stable or right stable component. A semi-stable component is *trivial* if it contains only one module.

The following shows that the semi-stable parts of Γ_A carry sufficient information of Γ_A.

3.1. Proposition. *Let A be an artin algebra. Then*

(1) *All but finitely many modules in Γ_A lie in some non-trivial semi-stable components of Γ_A.*

(2) *All but finitely many non-trivial semi-stable components of Γ_A are connected components of Γ_A.*

(3) *Each connected component of Γ_A is covered by finitely many semi-stable components of Γ_A.*

Proof. (1) Let $X \in \Gamma_A$. If X is left stable and not τ-periodic, then there exists $r \geq 0$ such that for all $n \geq r$, $\tau^n X$ is not a direct predecessor of any projective module in Γ_A. That is, the modules $\tau^n X$ with $n \geq r$ belong to the same left stable component of Γ_A. Dually if X is right stable, then either X is τ-periodic or there exists $s \geq 0$ such that the modules $\tau^{-m} X$ with $m \geq s$ belong to the same right stable component of Γ_A. Hence any τ-orbit in Γ_A contains at most finitely many modules which are not in any non-trivial semi-stable component of Γ_A. Moreover, if X is not in any non-trivial left stable component, then X is in the τ-orbit of a module which is either a projective module or a direct predecessor of a projective module. Thus there exist at most finitely τ-orbits in Γ_A which contain modules not in any non-trivial semi-stable component of Γ_A.

(2) Let Γ be a non-trivial semi-stable component of Γ_A. If Γ is not a connected component of Γ_A, then it contains a module which is either a direct predecessor of a projective module or a direct successor of an injective module in Γ_A. This completes the proof since (3) clearly follows from the first two statements.

Let Γ be a non-trivial semi-stable component of Γ_A. Then Γ is a translation quiver with the translation induced from the Auslander-Reiten translation in Γ_A.

3.2. Lemma. *Let A be an artin algebra, and let Γ be a non-trivial left stable component of Γ_A. Then Γ as a translation quiver is left stable and contains at most finitely many injective vertices.*

Proof. Let $X \in \Gamma$. Then X is not a projective module. If $\tau X \notin \Gamma$, then all direct predecessors and successors of X in Γ_A are not left stable modules. Thus Γ contains only one module X, which is a contradiction. Thus Γ is left stable as a translation quiver. Now let Y be an injective vertex in Γ which is not an injective module. Then Y has a direct successor in Γ_A which is not a left stable module. Hence there exists $n \geq 0$ such that $\tau^n Y$ is a direct predecessor of a projective module in Γ_A. Note that different injective vertices in Γ lie in different τ-orbits of Γ_A. Thus Γ has at most finitely many injective vertices.

As in immediate consequence of Theorem 2.5 and the above lemma, we can now describe the shapes of semi-stable components of Γ_A containing no oriented cycle.

3.3. Theorem. *Let A be an artin algebra, and let Γ be a non-trivial left stable component of Γ_A without oriented cycles. Then Γ contains a section Δ of non-Dynkin type such that Δ contains a unique sink and has no projective predecessor in Γ_A. Consequently Γ is isomorphic to the full subquiver of $\mathbf{Z}\Delta$ generated by the vertices (n, X) with $n \in \mathbf{Z}$ and $X \in \Delta$ such that $\mathrm{DTr}^n X \in \Gamma$.*

Proof. By Lemma 3.2, Γ is a left stable translation quiver with at most finitely many injective vertices. By Theorem 2.4, Γ has a section Σ with a unique sink. It is clear that there exists $t \geq 0$ such that for any $i \geq t$, $\tau^i \Sigma$ contains no direct predecessor of any projective module in Γ_A. Let $\Delta = \tau^t \Sigma$. Then Δ has no projective predecessor in Γ_A. Thus the predecessors of Δ in Γ_A are all in Γ. Using Proposition 2.7 in [**8**] and Lemma 1.7 in [**23**], we infer that Δ is not of Dynkin type.

We shall now describe the shapes of semi-stable components of Γ_A containing oriented cycles. First note that if a semi-stable component Γ contains a DTr-periodic module, then all modules in Γ are DTr-periodic. Hence we can apply the following result.

3.4. Theorem [30, 55]. *Let A be an artin algebra, and let Γ be a non-trivial semi-stable component of Γ_A containing a* DTr*-periodic module.*

(1) *If Γ is infinite, then Γ is a stable tube.*

(2) *If Γ is finite, then $\Gamma \cong \mathbf{Z}\Delta/G$, where Δ is a Dynkin quiver and G is an automorphism group of $\mathbf{Z}\Delta$.*

Remark. Hoshino considered a special case of the above result in [**32**]. He showed that if X is a module in Γ_A such that $DTrX = X$, then either the connected component of Γ_A containing X is a homogeneous stable tube or A is a local Nakayama algebra.

In the study of semi-stable components with oriented cycles, the following lemma plays a crucial role.

3.5. Lemma. *Let A be an artin algebra, and let Γ be a left stable component of Γ_A. Assume that*

$$\sigma : \qquad \cdots \to X_s \to X_{s-1} \to \cdots \to X_1 \to X_0$$

is an infinite sectional path in Γ containing infinitely many arrows of finite global left degree. Then

(1) *Each arrow in σ has trivial valuation.*

(2) *Each module in σ has at most two direct predecessors in Γ.*

(3) *σ is not contained in any double infinite sectional path in Γ.*

Proof. First of all (3) follows from Lemma 1.12.(2). We now claim that for any $i \geq 0$, the middle term of the almost split sequence ending with X_i has at most two left stable indecomposable summands including multiplicities. Assume that for some $t \geq 0$, the middle term of the almost split sequence ending with X_t admits a direct summand $Y \oplus Z \oplus X_{t+1}$ with Y, Z left stable. By assumption, there exists $k > t$ such that the arrow $X_{k+1} \to X_k$ has finite global left degree. Since $\tau^{n+1}X_k \to \tau^n X_{k+1}$ is of infinite left degree for any $n \geq 0$ by Lemma 1.5, there exists $m \geq 0$ such that $\tau^m X_{k+1} \to \tau^m X_k$ is of finite left degree. Thus $\tau^{m+k-t}X_{t+1} \to \tau^{m+k-t}X_t$ is of finite degree by Lemma 1.5 again. Since $\tau^{m+k-t}X_{t+1} \oplus \tau^{m+k-t}Y \oplus \tau^{m+k-t}Z$ is a summand of the middle term of the almost split sequence ending with $\tau^{m+k-t}X_t$, there exists an irreducible map

$$g : \tau^{m+k-t+1}X_t \to \tau^{m+k-t}Y \oplus \tau^{m+k-t}Z$$

of finite left degree. By Proposition 1.10, there exists an irreducible map

$$h : \tau^{m+k-t+1}Y \oplus \tau^{m+k-t+1}Z \to \tau^{m+k-t+1}X_t$$

of finite degree, which is a contradiction to Lemma 1.5. Thus what we claimed is true. Hence each X_i with $i \geq 0$ has at most two direct predecessors in Γ and each arrow $X_{i+1} \to X_i$ with $i > 0$ has trivial valuation. Let (d, d') be the valuation of the arrow $X_1 \to X_0$. Then $d = 1$. Assume that $d' > 1$. Then there exists an irreducible map from $X_1 \oplus X_1$ to X_0. By Lemma 1.12.(1), σ contains no arrow of finite global left degree, which is a contradiction. Hence $X_1 \to X_0$ also has trivial valuation. The proof is completed.

A valued translation quiver is said to be *smooth* if each arrow has trivial valuation and each vertex has at most two direct predecessors and at most two direct successors.

3.6. Theorem. *Let A be an artin algebra, and let Γ be a non-trivial left stable component of Γ_A containing no* DTr*-periodic module. If Γ contains an oriented cycle, then it is smooth and contains an infinite sectional path*

$$\cdots \to \mathrm{DTr}^r X_s \to \cdots \to \mathrm{DTr}^r X_2 \to \mathrm{DTr}^r X_1 \to X_s \to \cdots \to X_2 \to X_1,$$

where $r > s \geq 1$, and the X_i are not all stable and meet each DTr*-orbit in Γ exactly once.*

Proof. Assume that Γ contains an oriented cycle. Let

$$Y_1 \to Y_2 \to \cdots \to Y_s \to Y_{s+1} \quad (*)$$

be a path in Γ of minimal positive length such that $Y_{s+1} = \tau^t Y_1$ for some $t \geq 0$. Then $t > 0$ since Γ contains no τ-periodic module and Γ_A contains no sectional oriented cycle [**13**]. Suppose that $Y_j = \tau^k Y_i$ with $1 \leq i < j \leq s$ and $k \in \mathbf{Z}$. Then we have paths

$$Y_i \to \cdots \to Y_j = \tau^k X_i$$

and

$$Y_j \to \cdots \to Y_s \to \tau^t Y_1 \to \cdots \to \tau^t Y_i = \tau^{t-k} Y_j$$

of length $< s$. This contradicts the minimality of the length of the path $(*)$ since either $k \geq 0$ or $t - k > 0$. Hence we have shown that the Y_i with

$1 \leq i \leq s$ belong to pairwise different τ-orbits. Let $r = s+t$ and $X_i = \tau^{i-1}Y_i$ for $1 \leq i \leq s$. Then $r > s$ and

$$\cdots \to \tau^r X_s \to \cdots \to \tau^r X_2 \to \tau^r X_1 \to X_s \to \cdots \to X_2 \to X_1 \quad (**)$$

is an infinite sectional path in Γ. For each $j \geq 0$, consider the subpath

$$\tau^{(j+1)r} X_1 \to \tau^{jr} X_s \to \cdots \to \tau^{jr} X_2 \to \tau^{jr} X_1 \qquad (***)$$

of the path $(**)$. Since $r > s$, there exists an oriented cycle from $\tau^{(j+1)r}X_1$ to $\tau^{(j+1)r}X_1$ containing only modules of form $\tau^n X_i$ with $jr \leq n \leq (j+1)r$. Hence the path $(***)$ contains an arrow of finite global left degree. Therefore the path $(**)$ contains infinitely many arrows of finite global left degree. By Lemma 3.5, each module in the path $(**)$ has at most two direct predecessors in Γ and each arrow has trivial valuation. Moreover at least one the of X_i is not stable since otherwise $(**)$ could be extended to a double infinite sectional path in Γ.

Assume that $\tau^p X_j$ with $p \in \mathbf{Z}$ is a module in Γ and Z is a direct predecessor of $\tau^p X_j$ in Γ. Let $q > 0$ be an integer such that $qr - p > 0$. Then $\tau^{qr-p}Z$ is a direct predecessor of $\tau^{qr}X_j$. Thus $\tau^{qr-p}Z$, and hence Z is in the τ-orbit of the X_i. Therefore all modules in Γ belong to the τ-orbits of the X_i since Γ is left stable. That is, the X_i constitute a complete set of representatives of the τ-orbits in Γ. Now it is easy to see that each module in Γ has at most two direct predecessors in Γ and each arrow in Γ has trivial valuation. This completes the proof.

A connected component of Γ_A is said to be *semiregular* if it does not contain both a projective module and an injective module. Hence a semiregular component itself is a semi-stable component of Γ_A. As a special case of the above result, we have the following.

3.7. Theorem. *Let A be an artin algebra, and let $\mathcal{C}$ be a semiregular component of Γ_A with an oriented cycle. Then $\mathcal{C}$ is a coray tube, a stable tube or a ray tube.*

Proof. First of all $\mathcal{C}$ is clearly infinite [**1**]. We need only consider the case where $\mathcal{C}$ contains no projective module. Assume that $\mathcal{C}$ is not a stable tube. Then $\mathcal{C}$ contains no τ-periodic module. Thus by Theorem 3.6, $\mathcal{C}$ is smooth and it contains an infinite sectional path

$$\cdots \to \tau^r X_s \to \cdots \to \tau^r X_2 \to \tau^r X_1 \to X_s \to \cdots \to X_2 \to X_1 \quad (*)$$

where $r > s \geq 1$, and the X_i are not all stable and meet each τ-orbit in $\mathcal{C}$ exactly once. For convenience, let $Z_{ks+j} = \tau^{kr}X_j$ for $k \geq 0$ and $1 \leq j \leq s$. We claim that for any $m, i > 0$, if $\tau^{-m}Z_i$ is defined, then $\tau^{-m}Z_{i+1}$ is also defined. Suppose that this is not true. Let $n > 0$ be the least such that $\tau^{-n}Z_{i_0}$ is defined, but $\tau^{-n}Z_{i_0+1}$ is not. Then $\tau^{-n+1}Z_i$ is defined for all $i \geq i_0$ and $\tau^{-n+1}Z_{i_0+1}$ is injective. Since $\mathcal{C}$ is smooth, $\tau^n Z_{i_0}$ has exactly one direct predecessor, say Y_0 in $\mathcal{C}$. Then there exists an irreducible monomorphism from $\tau^{-n+1}Z_{i_0}$ to Y_0. Thus Y_0 has exactly two distinct direct predecessors $\tau^{-n+1}Z_{i_0}$ and Y_1, and there exists an irreducible monomorphism from τY_0 to Y_1. Similarly Y_1 has exactly two distinct direct predecessors τY_0 and Y_2, and there exists an irreducible monomorphism from τY_1 to Y_2. Inductively we get an infinite sectional path

$$\cdots \to Y_{i+1} \to Y_i \to \cdots \to Y_2 \to Y_1 \to Y_0$$

in $\mathcal{C}$. By Lemma 1.12. (1), the infinite sectional path

$$\cdots \to \tau^{-n+1}Z_{i+1} \to \tau^{-n+1}Z_i \to \cdots \to \tau^{-n+1}Z_{i_0+1} \to \tau^{-n+1}Z_{i_0} \to Y_0$$

contains no arrow of finite global left degree, and hence neither does the path $(*)$. This is clearly a contradiction. Therefore our claim is true. Thus the X_i are all non-stable, and if $X_i = \tau^{n_i}I_i$ with I_i injective and $n_i \geq 0$, then $n_1 \leq n_2 \leq \cdots \leq n_s$. Hence $\mathcal{C}$ is a coray tube.

From Theorems 3.3 and 3.7, the shapes of semiregular Auslander-Reiten components are fairly well-described. However so far we have no affirmative answer to the following well-known problem.

Problem 1. *Let A be a connected artin algebra of infinite representation type. Does the Auslander-Reiten quiver of A necessarily have a semiregular component or at least two connected components*?

To conclude this section, we shall have a short discussion on the problem of which valued translation quivers can be realized as Auslander-Reiten components. First of all, Brenner has established a combinatorial criterion for a finite valued translation quiver to occur as an Auslander-Reiten quiver [**18**]. Secondly one knows that all coray tubes, stable tubes and ray tubes can be realized as semiregular components of Auslander-Reiten quivers [**22**].

Finally we consider $\mathbf{Z}\Delta$ with Δ a locally finite and symmetrizable valued quiver without oriented cycles. A combinatorial argument shows that $\mathbf{Z}\Delta$ is not realizable if Δ is of Dynkin or Euclidean type. On the other hand, if Δ is neither Dynkin nor Euclidean such that all but finitely many vertices in Δ have at most two neighbours and all but finitely many arrows have trivial valuation, then $\mathbf{Z}\Delta$ can be realized as a regular component of an Auslander-Reiten quiver [**21, 51**]. For the remaining cases, the problem is still open.

4. Non-semiregular Components

In this section we shall study almost split sequences in non-semiregular components of Γ_A. The result shows that an Auslander-Reiten component either is not too complicate or behaves fairly well.

For a module M in mod A, we denote by $\ell(M)$ its composition length.

4.1. Lemma. *Let A be an artin algebra, and let*

$$0 \to X \xrightarrow{f} \oplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence in $\operatorname{mod} A$ *with the Y_i indecomposable. If $\ell(X) > \ell(Y_i)$ for all $1 \le i \le r$, then any sectional path in Γ_A ending with Z contains no projective module.*

Proof. Assume that $\ell(X) > \ell(Y_i)$ for all $1 \le i \le r$. Let

$$Z_n \to Z_{n-1} \to \cdots \to Z_1 \to Z_0 = Z$$

be a sectional path in Γ_A with $n > 0$. Then $Z_1 \cong Y_k$ for some $1 \le k \le r$, and hence $\ell(X) > \ell(Z_1)$. So Z_1 is not projective. If $n > 1$, then $\ell(\tau Z_1) > \ell(Z_2)$, and hence Z_2 is not projective. Continuing this process, we conclude that all Z_i are not projective.

4.2. Lemma. *Let A be an artin algebra, and let $f : X \to \oplus_1^4 Y_i$ be an irreducible map in* $\operatorname{mod} A$*, where X is indecomposable and the Y_i are indecomposable non-projective. If*

$$2\ell(X) \ge \sum_{i=1}^{4} \ell(Y_i),$$

then X has no projective predecessor in Γ_A.

Proof. Suppose that $2\ell(X) \geq \sum_{i=1}^4 \ell(Y_i)$. Since $\ell(\tau Y_i) + \ell(Y_i) \geq \ell(X)$ for $1 \leq i \leq 4$, we have

$$\sum_{i=1}^{4} \ell(\tau Y_i) \geq 4\ell(X) - \sum_{i=1}^{4} \ell(Y_i) > \ell(X).$$

Thus X is non-projective. Let $\tau X \to W$ be an arrow in Γ_A. If $W \not\cong \tau Y_i$ for all $1 \leq i \leq 4$, then

$$\ell(\tau X) \geq \ell(W) + \sum_{1}^{4} \ell(\tau Y_i) - \ell(X) \geq \ell(W) + \sum_{1}^{4} (\ell(X) - \ell(Y_i)) - \ell(X) > \ell(W).$$

If $W \cong \tau Y_i$ for some i, say $W \cong \tau Y_1$, then

$$\ell(\tau X) \geq \sum_{1}^{4} \ell(\tau Y_i) - \ell(X) \geq \ell(W) + \sum_{2}^{4} (\ell(X) - \ell(Y_i)) - \ell(X) > \ell(W).$$

By Lemma 4.1, any sectional path in Γ_A ending with X contains no projective module. Moreover,

$$\ell(\tau X) \geq \sum_{1}^{4} \ell(\tau Y_i) - \ell(X) \geq \sum_{1}^{4} (\ell(X) - \ell(Y_i)) - \ell(X) \geq \ell(X).$$

Thus $2\ell(\tau X) \geq \sum_1^4 \ell(\tau Y_i)$. Similarly any sectional path in Γ_A ending with τX contains no projective module. The lemma now follows by induction.

4.3. Corollary. *Let A be an artin algebra, and let $f : X \to \oplus_1^4 Y_i$ an irreducible epimorphism in* mod A *with X and the Y_i indecomposable. Then X has no projective predecessor in Γ_A.*

We are ready to get our main result of this section.

4.4. Theorem. *Let A be an artin algebra, and let*

$$0 \to X \xrightarrow{f} \oplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence in mod A *with the Y_i indecomposable. Assume that X has a projective predecessor and Z has an injective successor in Γ_A.*

Then $r \leq 4$, and the equality occurs only when one of the Y_i is projective-injective and the others are neither.

Proof. Let $r \geq 4$. We first consider the case where $\ell(Z) \geq \ell(X)$. Then $2\ell(Z) \geq \sum_{i=1}^{r} \ell(Y_i)$. By the dual of Lemma 4.2, one of the Y_i, say Y_r is injective. Then $\ell(X) > \sum_{i=1}^{r-1} \ell(Y_i)$. Thus $r = 4$ by Corollary 4.3. Moreover X is not projective since

$$\sum_{i=1}^{r-1} \ell(\tau Y_i) \geq 3\ell(X) - \sum_{i=1}^{r-1} \ell(Y_i) > \ell(X).$$

Assume that Y_4 is not projective. Then the modules τY_i with $1 \leq i \leq 4$ are not injective. Since X has an injective successor in Γ_A, we have $\ell(\tau X) > \ell(X)$ by the dual of Lemma 4.2. Hence τX has no projective predecessor in Γ_A by Lemma 4.2. Thus there exists a sectional path in Γ_A ending with X which contains a projective module. By Lemma 4.1, there exists an arrow $\tau X \to Y$ such that $\ell(\tau X) < \ell(Y)$. If $Y \not\cong \tau Y_i$ for all $1 \leq i \leq 4$, then $\sum_1^4 \ell(\tau Y_i) < \ell(X)$. This contradicts the dual of Lemma 4.2 since X has an injective successor in Γ_A. Thus $Y \cong \tau Y_i$ for some $1 \leq i \leq 4$. However since

$$\ell(\tau Y_1) + \ell(\tau Y_2) \geq 2\ell(X) - (\ell(Y_1) + \ell(Y_2)) > \ell(X),$$

we have $\ell(\tau X) > \ell(\tau Y_3) + \ell(\tau Y_4)$. Similarly we have $\ell(\tau X) > \ell(\tau Y_1)$ since $\ell(\tau Y_2) + \ell(\tau Y_3) > \ell(X)$ and $\ell(\tau X) > \ell(\tau Y_2)$ since $\ell(\tau Y_1) + \ell(\tau Y_3) > \ell(X)$. This contradiction shows that Y_4 is projective. Thus the theorem holds in this case. Dually we can show that the theorem holds in the case where $\ell(X) \geq \ell(Z)$.

Remark. (1) The above theorem also holds if we assume instead that X is DTr-periodic.

(2) It is well-known that if A is of finite representation type, then any indecomposable module has a projective predecessor and an injective successor in Γ_A. Hence the above result generalizes the Bautista-Brenner theorem [**10**].

5. Modules of Bounded Lengths in a Component

In this section we are concerned with the problem whether the number of modules in an Auslander-Reiten component of the same length is finite.

We will show that this is not true in general by an infinite DTr-orbit of modules of dimension four. Nevertheless it is still interesting to investigate in which Auslander-Reiten components the problem has an affirmative answer. For example, this is always the case for Auslander-Reiten components of hereditary algebras [**49, 57**] and tame algebras [**20**].

5.1. Lemma. *Let A be an artin algebra, and let Γ be a left stable component of Γ_A containing oriented cycles. Then for any module $X \in \Gamma$, the set $\{\mathrm{DTr}^n X; n \geq 0\}$ contains at most finitely many modules of any given length.*

Proof. Assume that Γ is infinite. If Γ is a stable tube, then the modules in Γ belong to a finite number of corays in Γ, and each of which contains at most finitely many modules of any given length by the Harada-Sai lemma and Proposition 1.1. Thus the lemma holds in this case. Otherwise by Theorem 3.6, Γ is smooth and contains an infinite sectional path

$$\sigma: \quad \cdots \to \tau^r X_s \to \cdots \to \tau^r X_2 \to \tau^r X_1 \to X_s \to \cdots \to X_2 \to X_1$$

where the X_i constitute a complete set of representatives of the τ-orbits in Γ. Note that the modules $\tau^n X_i$ with $1 \leq i \leq s$ and $n \geq 0$ are distributed into r sectional paths $\tau^j \sigma$, $j = 0, 1, \ldots, r-1$. The lemma follows again by the Harada-Sai Lemma and Proposition 1.1.

5.2. Proposition. *Let A be an artin algebra, and let Γ be a left stable component of Γ_A. If there exists a module $M \in \Gamma$ such that $\{\mathrm{DTr}^n M;\ n \geq 0\}$ contains an infinite number of modules of the same length, then Γ has a section of type A_∞.*

Proof. First of all there exists a constant $c > 1$ such that for any arrow $U \to V$ in Γ_A, $c^{-1}\ell(V) \leq \ell(U) \leq c\ \ell(V)$ (see [**48**, (2.1)]). Assume that $\{\tau^n M;\ n \geq 0\}$ contains an infinite number of modules of the same length for some $M \in \Gamma$. Then $\{\tau^n N;\ n \geq 0\}$ contains an infinite number of modules of the same length for any module $N \in \Gamma$. By Lemma 5.1, Γ contains no oriented cycle. By Theorem 3.3, Γ contains a section Δ with a unique sink X such that all predecessors of Δ in Γ_A are in Γ.

Assume that Δ is finite, say it has m modules. Let b be an integer such that $\ell(\tau^j X) = b$ for infinitely many $j > 0$. Note $\ell(\tau^j X) = b$ implies that $\ell(\tau^j Y) \leq c^m b$ for any $Y \in \Delta$. Thus there exist infinitely many $j > 0$ such that the modules in $\tau^j \Delta$ are of length $\leq c^m b$. Now choose a non-zero map

$\theta : P \to X$ with P a projective module in Γ_A. Then there exists an infinite chain

$$\cdots \to X_{i+1} \xrightarrow{f_i} X_i \to \cdots \to X_1 \xrightarrow{f_0} X_0 = X$$

of irreducible maps and homomorphisms $\theta_j : P \to X_j$ for all $j > 1$ such that $f = \theta_i f_{i-1} \cdots f_0$. Note that the X_i belong to Γ. Since Δ is finite, the modules X_i intersect $\tau^j \Delta$ for each $j \geq 0$. This contradicts the Harada-Sai Lemma. Thus Δ is infinite. By König's lemma, there exists an infinite path

$$\cdots \to Y_{i+1} \to Y_i \to \cdots \to Y_1 \to Y_0 = X$$

in Δ. By the Harada-Sai lemma and Lemma 1.3.(2), each arrow $Y_{i+1} \to Y_i$ has finite global left degree. Thus Δ is of type A_∞ by Lemmas 3.4 and 1.12.

As an immediate consequence we have the following.

5.3. Theorem. *Let A be an artin algebra, and let Γ be a stable component of Γ_A. If there exists a* DTr*-orbit in Γ which contains an infinite number of modules of the same length, then Γ is of shape* $\mathbf{Z}\mathrm{A}_\infty$.

Proof. Assume that Γ contains a module X whose τ-orbit contains an infinite number of modules of the same length. We may assume that $\{\tau^n X;\ n \leq 0\}$ contains an infinite number of modules of the same length. By the dual of Proposition 5.2, the right stable component of Γ_A containing Γ has a section of type A_∞. Consequently Γ has a section of type A_∞. Hence $\Gamma \cong \mathbf{Z}\mathrm{A}_\infty$.

We believe that the above result should hold in a more general context as follows.

Problem 2. *Let A be an artin algebra, and let Γ be a stable component of Γ_A. If Γ contains an infinite number of modules of the same length, is it necessarily of shape* $\mathbf{Z}\mathrm{A}_\infty$?

As another application of Proposition 5.2, we obtain the following theorem of Bautista and Coelho which generalizes the the related results in [**36, 41**].

5.4. Theorem [12]. *Let A be an artin algebra, and let $\mathcal{C}$ be a connected component of Γ_A in which all but finitely many* DTr*-orbits are periodic. Then $\mathcal{C}$ contains at most finitely many modules of any given length.*

Proof. Let Γ be a semi-stable, say left stable component of Γ_A contained in $\mathcal{C}$. Assume that Γ contains an infinite number of modules of the same length. We have seen that Γ is not a stable tube. Thus Γ has only finitely many τ-orbits by the assumption on $\mathcal{C}$. Hence Γ has a module X whose τ-orbit contains an infinite number of modules of the same length. By Proposition 5.2, X is right stable and the set $\{\tau^n X; n \leq 0\}$ contains an infinite number of modules of the same length. Note that there exists some $t \leq 0$ such that the modules $\tau^n X$ with $n \leq t$ belong to the same right stable component of Γ_A. By the dual of Proposition 5.2, this right stable component has a section of type A_∞, which is a contradiction. Therefore Γ contains at most finitely many modules of any given length. The result now follows from Proposition 3.1.

Recall that A is of *strongly unbounded representation type* if there exist infinitely many positive integers d such that there exist infinitely many modules of length d in Γ_A. Smalø has showed that A is of strongly unbounded representation type if Γ_A contains an infinite number of modules of the same length. The second Brauer-Thrall conjecture, which has been established for algebras over infinite perfect fields [**9, 16, 42**], states that a finite dimensional algebra over an infinite field is either of finite representation type or of strongly unbounded representation type.

As an immediate consequence of Theorem 5.4, we have the following.

5.5. Theorem. *Let A be an artin algebra. Assume that Γ_A has only finitely many* DTr*-orbits. Then A is not of strongly unbounded representation type. Consequently if in addition A is a finite dimensional algebra over an infinite perfect field, then A is of finite representation type.*

Proof. By assumption Γ_A has only finitely many connected components. By Theorem 5.4, each connected component has at most finitely many modules of the same length. Thus A is not of strongly unbounded representation type.

It is well-known that if A is of finite representation type, then every non-zero non-isomorphism between indecomposable modules is a sum of composites of irreducible maps [**6**]. We shall show that the converse is true for algebras over infinite perfect fields.

5.6. Theorem. *Let A be an artin algebra. Assume that every non-zero non-isomorphism between modules in Γ_A is a sum of composites of irreducible maps. Then A is not of strongly unbounded representation type. Consequently if in addition A is a finite dimensional algebra over an infinite perfect field, then A is of finite representation type.*

Proof. By assumption, every module in Γ_A is a successor of a projective module and a predecessor of an injective module. Thus Γ_A has only finitely many connected components. Let Γ be a semi-stable component of Γ_A without τ-periodic modules. By Theorem 3.3 and its dual, Γ contains oriented cycles. By Theorem 3.6 and its dual, Γ contains only finitely many τ-orbits. Hence by Proposition 3.1, all but finitely many τ-orbits in Γ_A are τ-periodic. The theorem follows now from Theorem 5.4.

In the preceding results we have to use the second Brauer-Thrall conjecture to deduce that A is of finite representation type. A direct proof will be useful to give an affirmative answer to the following.

Problem 3. *Let A be an artin algebra. Is A necessarily of finite representation type if either* (i) *there exist only finitely many* DTr*-orbits in Γ_A or* (ii) *every non-zero non-isomorphism between modules in Γ_A is a sum of composites of irreducible maps*?

We shall now conclude these notes with the promised example. In [**5**], Schulz found an Ω-bounded but not Ω-periodic module over a QF-algebra. A symmetrization of the algebra yields an infinite DTr-orbit of modules of bounded dimension.

5.7. Example [40]. Let λ be a complex number of multiplicative order α, where $\alpha \in \mathbf{N} \cup \{\infty\}$. Let R_α be a $\mathbf{C}$-algebra generated by x, y with relations

$$x^2 = y^2 = yx + \lambda xy = 0.$$

Let T_α be the trivial extension of R_α by $DR_\alpha = \mathrm{Hom}_{\mathbf{C}}(R_\alpha, \mathbf{C})$. If $\{a, b, c, d\}$ is the $\mathbf{C}$-basis of DR_α dual to the $\mathbf{C}$-basis $\{1, x, y, xy\}$ of R_α, then

$$\{1, x, y, xy, a, b, c, d\}$$

is a $\mathbf{C}$-basis of T_α with multiplication as follows:

	1	x	y	xy	a	b	c	d
1	1	x	y	xy	a	b	c	d
x	x	0	xy	0	0	a	0	$-\lambda c$
y	y	$-\lambda xy$	0	0	0	0	a	b
xy	xy	0	0	0	0	0	0	a
a	a	0	0	0	0	0	0	0
b	b	a	0	0	0	0	0	0
c	c	0	a	0	0	0	0	0
d	d	c	$-\lambda b$	a	0	0	0	0

Note T_α is a local algebra with $J(T_\alpha)^4 = 0$, where $J(T_\alpha)$ is the radical of T_α. For $i \in \mathbf{Z}$, let $M_i = (x + \lambda^i y)T_\alpha$. Then $\dim_{\mathbf{C}} M_i = 4$. Since $(x + \lambda^i y)(x + \lambda^{i+1} y) = 0$, M_{i+1} is in the kernel of the epimorphism from T_α to M_i by the left multiplication with $(x + \lambda^i y)$. A simple calculation of the dimensions shows that M_{i+1} is the kernel. Thus

$$0 \to M_{i+1} \to T_\alpha \to M_i \to 0$$

is an exact sequence. Hence $M_i = \Omega^i M_0$ for all $i \in \mathbf{Z}$. Note $DTr = \Omega^{2i}$ in this case. Thus the modules M_{2i} with $i \in \mathbf{Z}$ constitute a DTr-orbit.

By Theorem 5.3, the stable component of Γ_{T_α} containing the M_{2i} is either a stable tube or of shape $\mathbf{Z}\mathrm{A}_\infty$. Note that $\dim_{\mathbf{C}} J(T_\alpha) = 7$. By calculating the dimensions, we infer that the stable component containing M_0 is a regular component of Γ_{T_α}.

Let $0 \le i < j < \alpha$ be integers. Using the fact that $T_\alpha = \mathbf{C} + J(T_\alpha)$ and $J(T_\alpha)^4 = 0$, we deduce that $M_j(x + \lambda^{j+1} y)J(T_\alpha) = 0$, and that

$$M_i(x + \lambda^{j+1} y)J(T) = (\lambda^{j+1} - \lambda^{i+1})(xy)J(T_\alpha) \neq 0$$

since $xyd = a$. Thus M_i, M_j have different annihilators in T_α, and hence they are not isomorphic.

(1) If $\alpha = \infty$, then the modules M_i with $i \in \mathbf{Z}$ are pairwise not isomorphic. Hence the modules M_{2i} with $i \in \mathbf{Z}$ constitute an infinite DTr-orbit of modules of dimension four.

(2) Let $n \in \mathbf{N}$ and $\alpha = 2n$. Then $M_{2n} = M_0$ and the modules M_i with $0 \le i < 2n$ are pairwise not isomorphic. Thus M_0 is DTr-periodic of period n. So the regular component containing M_0 is a stable tube of

rank n. Note that the algebra T_{2n} has only one simple module. Therefore the rank of a stable tube in general is not bounded by any function of the number of simple modules. This is in contrast to the case where the rank of a generalized standard stable tube is at most $s+1$ with s the number of simple modules. Here an Auslander-Reiten component $\mathcal{C}$ is *generalized standard* if $\Re^{\infty}(X, Y) = 0$ for all modules X, Y in $\mathcal{C}$ (see [**53**]).

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