The bounded derived category of an algebra with radical squared zero

Raymundo Bautista and *Shiping Liu*

Introduction

Throughout this paper, k stands for a field. Let A be a finite dimensional k-algebra, and A-mod the category of finite dimensional left A-modules. The homological properties of A-mod are recorded in the derived category $D^b(A)$ of bounded complexes in A-mod. We want to study this category in the following aspects. First of all, since $D^b(A)$ is a Krull-Schmidt category, it is important to understand what the indecomposable objects are. Secondly, if A is of finite global dimension, then the Auslander-Reiten theory applies in $D^b(A)$; see [11, 12], and we would like to compute the almost split triangles and describe the shapes of the Auslander-Reiten components. In certain cases, this will enable us to determine if two given algebras are derived equivalent or not. Finally, the complexity of $D^b(A)$ is measured by its type which is finite, discrete, tame, or wild; see [5, 9, 10, 22]. In case A is hereditary, $D^b(A)$ is well-understood; see [12]. Moreover, if A is a gentle algebra, then the indecomposable objects in $D^b(A)$ and the type of $D^b(A)$ are explicitly described in [5]. The aim of this paper is to study $D^b(A)$ in case A is elementary (that is, all simple modules are one dimensional) with radical squared zero. Our strategy is to find a proper covering of the ordinary quiver of A and then to study $D^b(A)$ in terms of the derived category of the bounded complexes of finite dimensional representations of the covering. Note that an elementary algebra with radical squared zero is the Koszul dual of the path algebra of its ordinary quiver. In this connection, our technique can be viewed as a combination of the covering theory [6] and the Koszul duality [4].

1. The minimal gradable covering of a quiver

A quiver (or oriented graph) Q consists of a set Q_0 of vertices and a set Q_1 of arrows between vertices, and the *type* of Q is its underlying graph. If $\alpha : a \to b$ is an arrow in Q, we say that α *starts at* a and *ends in* b and write $a = s(\alpha)$, $b = e(\alpha)$. For a vertex a in Q, denote by a^+ the set of arrows starting at a and by a^- the set of arrows ending in a. One says that Q is *locally finite* if a^+ and a^- are both finite for all $a \in Q_0$. A *path* p of positive length r in Q is a formal product $p = \alpha_1 \cdots \alpha_r$ with $\alpha_i \in Q_1$ such that $s(\alpha_i) = e(\alpha_{i-1})$, for all $0 < i \leq r$. Such a path p is called an *oriented cycle* if $s(\alpha_1) = e(\alpha_r)$.

To each vertex a, one associates a *trivial path* ε_a with $s(\varepsilon_a) = e(\varepsilon_a)$ which is of length 0 by convention. Furthermore, for each $\alpha \in Q_1$, we introduce a formal inverse α^{-1} with $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. A *walk* w in Q is a formal product $w = c_1 c_2 \cdots c_r$ with $r > 0$, where c_i is a trivial path, an arrow or the inverse of an arrow such that $s(c_i) = e(c_{i-1})$ for all $1 \le i < r$. In this case, we write $s(w) = s(c_1)$ and $e(w) = e(c_r)$, and we say that w is a *walk from* $s(w)$ *to* $e(w)$. If $w = c_1 \cdots c_r$ and $w' = c_1' \cdots c_s'$ are walks such that $e(w) = s(w')$, then $ww' = c_1 \cdots c_r c_1' \cdots c_s'$ is a walk, called the *composite* of w and w'. A walk w in Q is called *closed* if $s(w) = e(w)$; *reduced* if w is either a trivial path, or $w = c_1 \cdots c_r$ with $c_i \in Q_1$ or $c_i^{-1} \in Q_1$ such that $c_{i+1} \ne c_i^{-1}$ for all $1 \le i < r$; and a *cycle* if w is non-trivial, reduced and closed. The *degree* $\partial(w)$ of a walk w is defined as follows. We first define $\partial(w) = 0, 1$, or -1 in case w is a trivial path, an arrow, or the inverse of an arrow respectively, and then extend this definition to all walks in Q by $\partial(uv) = \partial(u) + \partial(v)$ whenever u, v are walks with $e(u) = s(v)$. In particular, a path is a walk whose degree is equal to its length. The set of walks in Q will be denoted by $W(Q)$. One says that Q is *connected* if, for any $x, y \in Q_0$, there exists some $w \in W(Q)$ with $s(w) = x$ and $e(w) = y$.

For vertices x, y in Q, we denote by $Q(x, y)$ the set of paths in Q from x to y, by $Q_{\le 1}(x, y)$ the set of paths from x to y of length less than or equal to 1, and by $Q_1(x, y)$ the set of arrows from x to y. A quiver Δ is a *subquiver* of Q if $\Delta_i \subseteq Q_i$ for $i = 0, 1$. A subquiver Δ of Q is *full* if $\Delta_1(x, y) = Q_1(x, y)$ for all $x, y \in \Delta_0$; and *convex* if a path in Q lies entirely in Δ whenever its starting point and end-point lie in Δ.

A *quiver-morphism* $\phi : Q' \to Q$ consists of two maps $\phi_0 : Q_0' \to Q_0$ and $\phi_1 : Q_1' \to Q_1$ such that $\phi_1(Q_1'(a, b)) \subseteq Q_1(\phi_0(a), \phi_0(b))$ for all $a, b \in Q_0'$. In this case, ϕ induces naturally a map from $W(Q')$ to $W(Q)$, denoted again by ϕ, such that $\partial(\phi(w)) = \partial(w)$ for all $w \in W(Q')$.

Furthermore, a quiver-morphism $\pi = (\pi_0, \pi_1) : \widetilde{Q} \to Q$ is called a *covering* if π_0 is surjective, and for each vertex a in $\widetilde{Q}$, the map π_1 induces two bijections $a^+ \to (\pi_0(a))^+$ and $a^- \to (\pi_0(a))^-$. In this case, an automorphism σ of $\widetilde{Q}$ is called a π-*automorphism* if σ makes the diagram

$$\begin{tikzcd} \widetilde{Q} \arrow[r, "\sigma"] \arrow[d, "\pi"] & \widetilde{Q} \arrow[d, "\pi"] \\ Q \arrow[r, equal] & Q \end{tikzcd}$$

commutative. We denote by $\mathrm{Aut}_\pi(\widetilde{Q})$ the group of π-automorphisms of $\widetilde{Q}$.

1.1. Definition. A quiver Q is called *gradable* if every closed walk in Q is of degree zero.

Remark. A quiver without cycles is evidently gradable. On the other hand, a gradable quiver contains no oriented cycle.

Let Q be a gradable quiver. Given $x, y \in Q_0$, all possible walks in Q from x to y have the same degree which we denote by $d(x, y)$. Defining $x \sim y$ provided that $d(x, y) = 0$ yields an equivalence relation $\sim$ on Q_0. The equivalence classes in $Q_0/\sim$ are called the *grading classes* of Q_0. Indeed, we may grade Q_0 in the following way. Fix arbitrarily $a \in Q_0$. For each $n \in \mathbb{Z}$, let $Q^n(a)$ denote the set of vertices x such that $d(a, x) = n$. Clearly, the classes in $Q_0/\sim$ are precisely the non-empty $Q^n(a)$ with $n \in \mathbb{Z}$.

1.2. Lemma. *Let Q be a connected gradable quiver with $a \in Q_0$.*
(1) *The set Q_0 is the disjoint union of the $Q^n(a)$ with $n \in \mathbb{Z}$.*
(2) *If $x \in Q^m(a)$, then $y \in Q^n(a)$ if and only if $d(x, y) = n - m$.*
(3) *If $Q^m(a) \neq \emptyset$ and $Q^n(a) \neq \emptyset$, then $Q^i(a) \neq \emptyset$ for all i between m and n.*
(4) *If $b \in Q_0$ with $d(a, b) = s$, then $Q^n(b) = Q^{n+s}(a)$ for all $n \in \mathbb{Z}$.*

Proof. Since Q is connected, Q_0 is the union of the $Q^n(a)$, $n \in \mathbb{Z}$, and since Q is gradable, the $Q^n(a)$ are pairwise disjoint. Thus (1) follows. Moreover, it is easy to see that $d(x, z) = d(x, y) + d(y, z)$ for all $x, y, z \in Q_0$, from which (2) and (4) follow trivially. For proving (3), we assume that $m < n$ and $Q^m(a) \neq \emptyset$ and $Q^n(a) \neq \emptyset$. We shall proceed by induction on $r = n - m$. If $r = 0$, then there is nothing to prove. Suppose that $r > 0$ and that (3) holds for $r-1$. Let $x \in Q^m(a)$ and $y \in Q^n(a)$. Then there exists a walk $w = c_1 \cdots c_t$ of degree r from x to y, where c_i or c_i^{-1} is an arrow in Q. Let s be the minimal integer between 1 and t such that $\partial(c_1 \cdots c_s) > 0$. Then $\partial(c_1 \cdots c_s) = 1$. Thus $d(x, s(c_s)) = 0$ and c_s is an arrow. Therefore, $e(c_s) \in Q^{m+1}(a)$. By the induction hypothesis, $Q^i(a) \neq \emptyset$, for any $m + 1 \leq i \leq n$. The proof of the lemma is completed.

The universal covering of a quiver is clearly gradable since it has no cycle. Next, we shall find a minimal such covering.

1.3. Theorem. *Let Q be a connected quiver. Then there exists a connected gradable quiver $\widetilde{Q}$ and a quiver-covering $\pi : \widetilde{Q} \to Q$ which acts injectively on each grading class of $\widetilde{Q}$.*

Proof. Fix arbitrarily a vertex x in Q. Let $W(Q, x)$ be the set of walks w in Q with $s(w) = x$. For $u, v \in W(Q, x)$, we define $u \sim v$ provided that $e(u) = e(v)$ and $\partial(u) = \partial(v)$. This is clearly an equivalence relation on $W(Q, x)$. For $u \in W(Q, x)$, let $[u] = \{w \in W(Q, x) \mid w \sim u\}$. Now we define a quiver $\widetilde{Q} = (\widetilde{Q}_0, \widetilde{Q}_1)$, where $\widetilde{Q}_0$ is the set of the classes $[u]$ with $u \in W(Q, x)$. For $\mu, \nu \in \widetilde{Q}_0$, if α is an arrow in Q such that $u \in \mu$ and $v \in \nu$ with $v = u\alpha$, then we draw an unique arrow $\alpha_{\mu\nu} : \mu \to \nu$, called the arrow from μ to ν *induced* from α. We define $\widetilde{Q}_1$ to be the set of arrows induced from the arrows in Q.

For $\mu \in \widetilde{Q}_0$, let $\pi_0(\mu) = e(u)$ with $u \in \mu$. Since Q is connected, the map $\pi_0 : \widetilde{Q}_0 \to Q_0$ is surjective. If $\alpha_{\mu\nu} : \mu \to \nu$ is an arrow in $\widetilde{Q}$ induced from $\alpha \in Q_1$, we then define $\pi_1(\alpha_{\mu\nu}) = \alpha$. Clearly $\pi = (\pi_0, \pi_1)$ is a quiver-morphism from $\widetilde{Q}$ to Q. Let μ be a vertex in $\widetilde{Q}$ and write $a = \pi_0(\mu)$. For $\alpha \in a^+$, choose $u \in \mu$ and let $\nu = [u\alpha]$. Then $\alpha_{\mu\nu}$ is an arrow from μ to ν such that $\pi_1(\alpha_{\mu\nu}) = \alpha$. Moreover, if $\alpha_{\mu\nu'} : \mu \to \nu'$ is another arrow induced from α, then there exists $u' \in \mu$ such that $u'\alpha \in \nu'$. Since $u \sim u'$, we have $u\alpha \sim u'\alpha$. Hence $\nu' = \nu$. This shows that π_1 induces a bijection from μ^+ onto a^+. Similarly, we see that π_1 induces a bijection from μ^- onto a^-. Therefore, $\pi : \widetilde{Q} \to Q$ is a quiver-covering.

Now let $\mu, \nu \in \widetilde{Q}_0$. Choose $u \in \mu$ and $v \in \nu$, and set $a = \pi_0(\mu)$ and $b = \pi_0(\nu)$. Since Q is connected, there exists a walk $w = c_1 \cdots c_s$ in Q from a to b, where c_i is an arrow or the inverse of an arrow in Q. By the definition of $\widetilde{Q}_1$, we see that w induces a walk ρ from μ to ν such that $\pi_1(\rho) = w$. This shows that $\widetilde{Q}$ is connected. Moreover, if ρ is closed, that is $u \sim v$, then $\partial(u) = \partial(v)$. Hence $\partial(w) = 0$, and consequently $\partial(\rho) = 0$. This shows that $\widetilde{Q}$ is gradable. Finally, assume that $\partial(\rho) = 0$ and $a = b$. Then $e(u) = e(v)$ and $\partial(u) = \partial(v)$. Hence $u \sim v$, that is, $\mu = \nu$. This proves that the restriction of π_0 to each grading class of $\widetilde{Q}_0$ is injective. The proof of the theorem is completed.

The covering as stated Theorem 1.3 has certain universal property.

1.4. Theorem. *Let Q be a connected quiver with $\pi : \widetilde{Q} \to Q$ a quiver-covering as stated in Theorem 1.3. Let $\phi : \bar{Q} \to Q$ be a quiver-covering with $\bar{Q}$ connected and gradable. If $x^* \in \bar{Q}_0$ and $y^* \in \widetilde{Q}_0$ such that $\phi(x^*) = \pi(y^*)$, then there exists an unique quiver-covering $\psi : \bar{Q} \to \widetilde{Q}$ which sends x^* to y^* and makes the following diagram commutative*:

$$\begin{tikzcd} \bar{Q} \arrow[r, dotted, "\psi"] \arrow[d, "\phi"'] & \widetilde{Q} \arrow[d, "\pi"] \\ Q \arrow[r, equal] & Q \end{tikzcd}$$

Proof. Assume that $x^* \in \bar{Q}_0$ and $y^* \in \widetilde{Q}_0$ such that $\phi(x^*) = \pi(y^*)$. Let x be a vertex in $\bar{Q}$. Choose a walk u in $\bar{Q}$ from x^* to x. Then there exists an unique walk v in $\widetilde{Q}$ with $s(v) = y^*$ such that $\pi(v) = \phi(u)$. We define $\psi_0(x) = e(v)$. Note that $\partial(v) = \partial(\pi(v)) = \partial(\phi(u)) = \partial(u)$. Let u_1 be another walk in $\bar{Q}$ from x^* to x and v_1 a walk in $\widetilde{Q}$ with $s(v_1) = y^*$ such that $\pi(v_1) = \phi(u_1)$. Then we also have $\partial(v_1) = \partial(u_1)$ and, since $\bar{Q}$ is gradable, we get $\partial(u_1) = \partial(u)$. Hence $\partial(v^{-1}v_1) = 0$. That is, $e(v)$ and $e(v_1)$ lie in the same grading class of $\widetilde{Q}_0$. Moreover, $\pi(e(v)) = e(\pi(v)) = \phi(x)$, and $\pi(e(v_1)) = \phi(x)$. Since the action of π on each grading class is injective, we get $e(u_1) = e(u)$. This shows that ψ_0 is well-defined, and $\psi_0(x^*) = y^*$ by definition. If $\alpha : x \to y$ is an arrow in $\bar{Q}$, then there exists an unique $\beta \in \widetilde{Q}_1$ with $s(\alpha) = e(v)$ and $\pi(\beta) = \phi(\alpha)$. Since $\pi(v\beta) = \phi(u\alpha)$, one gets $\psi_0(y) = e(\alpha)$. Now we define $\psi_1(\alpha) = \beta$. This yields a quiver-morphism $\psi = (\psi_0, \psi_1)$ from $\bar{Q}$ to $\widetilde{Q}$ making the diagram stated in the theorem commutative. Since ϕ and π are both coverings, we deduce easily that

ψ is a covering. Finally, the uniqueness of ψ follows from a routine verification. The proof of the theorem is completed.

REMARK. It follows from Theorem 1.4 that a covering $\pi : \widetilde{Q} \to Q$ as stated in Theorem 1.3 is unique up to isomorphism. Thus we may call the covering morphism π, as well as the quiver $\widetilde{Q}$, the *minimal gradable covering* of Q. It is clear that Q is gradable if and only if π is an isomorphism.

1.5. LEMMA. *Let $\pi : \widetilde{Q} \to Q$ be the minimal gradable covering of a finite connected quiver Q, and let a be a vertex in $\widetilde{Q}$.*

(1) *The set $\widetilde{Q}^n(a)$ is finite, for all n.*

(2) *If Q is not gradable, then $\widetilde{Q}^n(a)$ is not empty for any n.*

(3) *If $x \in \widetilde{Q}^n(a)$ and $y \in \widetilde{Q}^{n+1}(a)$ for some n, then π induces a bijection from $\widetilde{Q}_1(x, y)$ onto $Q_1(\pi(x), \pi(y))$.*

(4) *If $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$ such that $\sigma(\widetilde{Q}^s) \cap \widetilde{Q}^t \neq \emptyset$ for some integers s, t, then $\sigma(\widetilde{Q}^{n+s}) = \widetilde{Q}^{n+t}$ for all n.*

Proof. For each $n \in \mathbb{Z}$, by Theorem 1.3, π induces an injection $\widetilde{Q}^n(a) \to Q$. In particular, $\widetilde{Q}^n(a)$ is finite. Assume now that there exists a closed walk w in Q of degree $s \neq 0$. Let $b = s(w)$ and $x \in \widetilde{Q}^t(a)$ such that $\pi(x) = b$. For each $m \in \mathbb{Z}$, there exists a walk u_m in $\widetilde{Q}$ with $s(u_m) = x$ and $\pi(u_m) = w^m$. Then $\partial(u_m) = \partial(w^m) = ms$. By Lemma 1.2(2), $e(u_m) \in \widetilde{Q}^{t+ms}$. Now it follows from Lemma 1.2(3) that $\widetilde{Q}^n(a) \neq \emptyset$, for any $n \in \mathbb{Z}$.

Next, let $(x, y) \in \widetilde{Q}^n(a) \times \widetilde{Q}^{n+1}(a)$ for some n. Being a covering, π induces an injection $\pi_1 : \widetilde{Q}_1(x, y) \to Q_1(\pi(x), \pi(y))$. If $\alpha \in Q_1(\pi(x), \pi(y))$, then there exists $\beta : x \to z$ in $\widetilde{Q}_1$ such that $\pi(\beta) = \alpha$. Note that $z \in \widetilde{Q}^{n+1}(a)$ and $\pi(z) = \pi(y)$. Thus $z = y$, and hence $\beta \in \widetilde{Q}_1(x, y)$. That shows that the map $\pi_1 : \widetilde{Q}_1(x, y) \to Q_1(\pi(x), \pi(y))$ is surjective.

Finally, let $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$ such that $\sigma(x) = y$ with $x \in \widetilde{Q}^s$ and $y \in \widetilde{Q}^t$. Let $\tilde{u}$ be a walk in $\widetilde{Q}$ from x to y. By Lemma 1.2(2), $\tilde{u}$ is of degree $t - s$. Let n be any integer and $z \in \widetilde{Q}^n(a)$. Choose a walk $\tilde{v}$ in $\widetilde{Q}$ from z to x. Then $\sigma(\tilde{v})$ is a walk from $\sigma(z)$ to y. Thus $\tilde{v}\tilde{u}\sigma(\tilde{v})^{-1}$ is a walk of degree $t - s$ from z to $\sigma(z)$. By Lemma 1.2(2), $\sigma(z) \in \widetilde{Q}^{n+t-s}(a)$. This implies that $\sigma(\widetilde{Q}^n(a)) \subseteq \widetilde{Q}^{n+t-s}(a)$, for all integers n. Since σ is an automorphism, $\sigma(\widetilde{Q}^n(a)) = \widetilde{Q}^{n+t-s}(a)$, for all integers n. Replacing n by $n + s$, we get $\sigma(\widetilde{Q}^{n+s}(a)) = \widetilde{Q}^{n+t}(a)$. The proof of the lemma is completed.

It is easy to see that a non-gradable quiver contains cycles of positive degree. This observation leads to the following definition.

1.6. DEFINITION. Let Q be a finite connected quiver. The *grading period* of Q is a non-negative integer r such that $r = 0$ in case Q is gradable, and otherwise, r is the minimal degree among the positive degrees of closed walks in Q.

1.7. Lemma. *Let $\pi : \widetilde{Q} \to Q$ be the minimal gradable covering of a finite connected quiver Q of grading period r, and let $a \in \widetilde{Q}_0$. The following conditions are equivalent for integers s, t :*

(1) $s \equiv t \,(\mathrm{mod}\, r)$.

(2) $\pi(\widetilde{Q}^{n+s}(a)) = \pi(\widetilde{Q}^{n+t}(a))$, *for all integers n.*

(3) $\pi(\widetilde{Q}^{n+s}(a)) \cap \pi(\widetilde{Q}^{n+t}(a)) \neq \emptyset$, *for some integer n.*

(4) $\sigma(\widetilde{Q}^{n+s}(a)) = \widetilde{Q}^{n+t}(a)$ *for some $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$ and some integer n.*

Proof. Let $(x, y) \in \widetilde{Q}^{n+t}(a) \times \widetilde{Q}^{n+s}(a)$ for some n such that $\pi(x) = \pi(y)$. By Theorem 1.4, there exists a π-automorphism σ of $\widetilde{Q}$ such that $\sigma(x) = y$. By Lemma 1.5(4), $\sigma(\widetilde{Q}^{n+s}(a)) = \widetilde{Q}^{n+t}(a)$. This proves that (3) implies (4). Moreover, if there exists some $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$ such that $\sigma(\widetilde{Q}^{m+s}(a)) = \widetilde{Q}^{m+t}(a)$ for some m. By Lemma 1.5(4), $\sigma(\widetilde{Q}^{n+s}(a)) = \widetilde{Q}^{n+t}(a)$ for all n. Since σ is a π-automorphism, $\pi(\widetilde{Q}^{n+s}(a)) = \pi(\widetilde{Q}^{n+t}(a))$ for all n. Since (2) implies trivially (3), we see that (2), (3), and (4) are equivalent.

Let w be a closed walk in Q of degree r. Choose a vertex b in $\widetilde{Q}$, say $b \in \widetilde{Q}^m(a)$ for some m, such that $\pi(b) = s(w)$. Let $\widetilde{w}$ be a walk in $\widetilde{Q}$ with $s(\tilde{w}) = b$ such that $\pi(\tilde{w}) = w$. Setting $c = e(\widetilde{w})$, we have $\pi(c) = \pi(b)$. Moreover, $c \in \widetilde{Q}^{m+r}$ since $\partial(\widetilde{w}) = \partial(w)$. That is $\pi(\widetilde{Q}^m) \cap \pi(\widetilde{Q}^{m+r}) \neq \emptyset$. By what we have shown, $\pi(\widetilde{Q}^n(a)) = \pi(\widetilde{Q}^{n+r}(a))$, for all integers n. If $s = t + rq$ for some integer q, then $\pi(\widetilde{Q}^s(a)) = \pi(\widetilde{Q}^{t+rq}(a)) = \widetilde{Q}^t$. That is, (1) implies (3). Suppose that $s \not\equiv t \,(\mathrm{mod}\, r)$. If $r = 0$, then Q is gradable. Hence π is an isomorphism. In particular, (2) does not hold. Assume that $r > 0$ but (2) holds. In particular, $\pi(\widetilde{Q}^0(a)) = \pi(\widetilde{Q}^{s-t}(a))$. Write $s - t = rq + r_0$ with $0 < r_0 < r$. Then $\pi(\widetilde{Q}^0(a)) = \pi(\widetilde{Q}^{rq+r_0}(a)) = \pi(\widetilde{Q}^{r_0}(a))$. Let $\pi(a) = \pi(x_0)$ for some $x_0 \in \pi(\widetilde{Q}^{r_0}(a))$. Choose a walk w_0 from a to x_0 in $\widetilde{Q}$. Then w_0 is of degree r_0, and consequently, $\pi(w_0)$ is a closed walk in Q of positive degree r_0, a contradiction. This proves that (2) implies (1). The proof of the lemma is completed.

In the sequel, we shall consider the following infinite graphs

$$\mathbb{A}_\infty : \quad \circ \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \cdots$$

$$\mathbb{A}_\infty^\infty : \quad \cdots \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \cdots$$

Moreover, we recall that a finite quiver is *wild* if it is of neither Dynkin nor Euclidean type.

1.8. Proposition. *Let $\pi : \widetilde{Q} \to Q$ be the minimal gradable covering of a finite connected quiver Q.*

(1) *If Q is non-gradable of type $\widetilde{\mathbb{A}}_n$ with $n \geq 1$, then $\widetilde{Q}$ is of type $\mathbb{A}_\infty^\infty$.*

(2) *If Q is wild, then $\widetilde{Q}$ contains a finite wild subquiver.*

Proof. First we define the weight of a vertex in a quiver to be the sum of the number of arrows starting at or ending in the vertex. Given any $x \in \widetilde{Q}_0$, we see that x and $\pi(x)$ have the same weight since π is a covering. Let Q be non-gradable of type $\widetilde{\mathbb{A}}_n$. Then every vertex in Q has weight two, and so does every vertex in $\widetilde{Q}$. Being infinite by Lemma 1.5(2), $\widetilde{Q}$ is of type $\mathbb{A}_{\infty}^{\infty}$. This proves (1). Suppose now that Q is wild. Then $\widetilde{Q}$ has a vertex a of weight greater than two. If Q is gradable, then $\widetilde{Q} \cong Q$. Otherwise, Q has positive grading period r. We deduce from Lemma 1.7 that $\pi(\widetilde{Q}^0(a)) = \pi(\widetilde{Q}^{nr}(a))$, for all $n \in \mathbb{Z}$. Thus $\pi^-(a)$ contains infinitely many vertices, each of them has weight greater than two. Therefore, $\widetilde{Q}$ has a finite wild subquiver. The proof of the proposition is completed.

2. Representations of Quivers

Throughout this section, Q stands for a connected locally finite quiver which is interval-finite, that is, $Q(a, b)$ is finite for all $a, b \in Q_0$. Recall that a k-*representation* M of Q consists of a family of k-spaces $M(x)$ with $x \in Q_0$, and a family of k-linear maps $M(\alpha) : M(x) \to M(y)$ with $\alpha : x \to y \in Q_1$. If M is non-zero, the *support* $\mathrm{Supp}(M)$ of M is the set of vertices x in Q for which $M(x) \neq 0$. We say that M is *locally finite dimensional* if $\dim_k M(x)$ is finite for all $x \in Q_0$, *finite dimensional* if $\sum_{x \in Q_0} \dim_k M(x)$ is finite. A morphism $f : M \to N$ of k-representations of Q consists of a family of k-linear maps $f(x) : M(x) \to N(x)$ with $x \in Q_0$ such that $M(\alpha)f(y) = f(x)N(\alpha)$ for all arrows $\alpha : x \to y$ in Q. The k-representations of Q form a hereditary abelian k-category, denoted as $\mathrm{Rep}(Q)$. The full subcategory of $\mathrm{Rep}(Q)$ of locally finite dimensional representations is denoted by $\mathrm{rep}(Q)$, and that of finite dimensional ones is denoted by $\mathrm{rep}^b(Q)$. On the other hand, the *path algebra* kQ (not necessarily with an identity) of Q over k has as k-basis the set of paths in Q and multiplication induced from the concatenation of the paths. We see that kQ has as a complete set of pairwise orthogonal primitive idempotents the set of trivial paths in Q and that kQ has an identity if and only if Q is finite. It is well known that $\mathrm{Rep}(Q)$ is equivalent to the category of right kQ-modules. In this connection, we shall apply some module theoretic notions to the k-representations of Q without further explanations.

To each vertex a in Q, we associate an indecomposable k-representation P_a of Q which is defined as follows: for a vertex x, the k-space $P_a(x)$ has as a basis the set of paths from a to x; and for an arrow $\alpha : x \to y$, the k-linear map $P_a(\alpha) : P_a(x) \to P_a(y)$ sends every path p to $p\alpha$. Since $P_a = \varepsilon_a(kQ)$ where ε_a is the trivial path at a, we see that P_a is a projective object in the abelian category $\mathrm{Rep}(Q)$. Dually, we define an indecomposable k-representation I_a of Q as follows: for $x \in Q_0$, the k-space $I_a(x)$ has as a basis the set of paths in Q from x to a; and for $\alpha : x \to y \in Q_1$, the k-linear map $I_a(\alpha) : I_a(x) \to I_a(y)$ sends every path of the form αq to q and vanishes on the paths which do not factor through α. In order to show that the I_a with $a \in Q_0$ are injective objects in $\mathrm{Rep}(Q)$, we recall that the tensor product $M \otimes_k V$ of an object M in $\mathrm{Rep}(Q)$

and a k-vector space V is defined so that $(M \otimes_k V)(a) = M(a) \otimes_k V$ for $a \in Q_0$ and $(M \otimes_k V)(\alpha) = M(\alpha) \otimes_k \mathbf{1}_V$ for $\alpha \in Q_1$.

2.1. LEMMA. *Let M be a k-representation of Q, and V be a k-vector space. For each vertex a in Q, there exists a k-linear isomorphism*

$$\phi_M : \mathrm{Hom}_{kQ}(M, I_a \otimes_k V) \to \mathrm{Hom}_k(M(a), V),$$

which is natural in M.

Proof. Let $a \in Q_0$. For $x \in Q_0$, we have $(I_a \otimes_k V)(x) = \oplus_{\rho \in Q(x,a)} (k\rho \otimes_k V)$. For $\rho \in Q(x,a)$, let $p_\rho : I_a(x) \to k\rho$ be the canonical projection, and let $q_\rho : k\rho \to I_a(a)$ be the k-linear isomorphism sending ρ to ε_a. We see then from the definition that $(V \otimes I_a)(\rho) = (p_\rho \otimes \mathbf{1}_V)(q_\rho \otimes \mathbf{1}_V)$. Given a morphism $f : M \to I_a \otimes_k V$ in $\mathrm{Rep}(Q)$, we define $\phi_M(f)$ to be the following composite:

$$M(a) \xrightarrow{f(a)} I_a(a) \otimes_k V \xrightarrow{e_V} V,$$

where e_V is such that $e_V(\lambda\varepsilon_a \otimes v) = \lambda v$, for $\lambda \in k$ and $v \in V$. It is evident that ϕ_M is k-linear. Assume that $\phi_M(f) = 0$. Since e_V is an isomorphism, $f(a) = 0$. Let x be an arbitrary vertex in Q. If $Q(x,a) = \emptyset$, then $f(x) = 0$. Otherwise, for any $\rho \in Q(x,a)$, we have

$$0 = M(\rho)f(a) = f(x)(I_a \otimes V)(\rho) = f(x)(p_\rho \otimes \mathbf{1}_V)(q_\rho \otimes \mathbf{1}_V).$$

Since q_ρ is an isomorphism, $f(x)(p_\rho \otimes \mathbf{1}_V) = 0$. This implies that $f(x) = 0$. Thus $f = 0$. That is, ϕ_M is a monomorphism.

For proving that ϕ_M is an epimorphism, let $g : M(a) \to V$ be a k-linear map. Define $f(a) = g e_V^{-1} : M(a) \to I_a(a) \otimes_k V$. For $x \in Q_0$, if $Q(x,a) = \emptyset$, then define $f(x) = 0$. Otherwise, $Q(x,a)$ is finite, since Q is interval-finite by hypothesis. In particular, $(I_a \otimes V)(x) = \prod_{\rho \in Q(x,a)} (k\rho \otimes V)$. Thus there exists a k-linear map $h : M(x) \to V \otimes I_a(x)$ such that $h(p_\rho \otimes \mathbf{1}_V) = M(\rho)g(a)(q_\rho^{-1} \otimes \mathbf{1}_V)$. Then

$$h(I_a \otimes V)(\rho) = h(p_\rho \otimes \mathbf{1}_V)(q_\rho \otimes \mathbf{1}_V) = M(\rho)g(a),$$

for every $\rho \in Q(x,a)$. Define now $f(x) = h$. It is now easy to see that the morphisms $f(x)$ with $x \in Q_0$ yield a morphism $f : M \to I_a \otimes_k V$ in $\mathrm{Rep}(Q)$ such that $\phi_M(f) = g$. Finally, it is easy to verify that ϕ_M is natural in M. The proof of the lemma is completed.

REMARK. For $a \in Q_0$, it follows from the above result that $I_a \otimes_k V$ is an injective object in the abelian category $\mathrm{Rep}(Q)$ for any k-vector space V. In particular, I_a itself is injective.

For the rest of this section, we assume that Q is gradable such that the grading classes are all finite. Fix a vertex a_0 in Q, and write $Q^n = Q^n(a_0)$ for all $n \in \mathbb{Z}$. By Lemma 1.2, Q_0 is the disjoint of the Q^n with $n \in \mathbb{Z}$, and each arrow in Q is of the form $x \to y$ with $(x,y) \in Q^n \times Q^{n+1}$ for some n. An object M in $\mathrm{Rep}(Q)$ is called *bounded-above* if there exists some integer r such that $M(x) = 0$ for $x \in Q^n$ with $n \geq r$. By Lemma 1.2(3), this notion does not depend on the choice of the vertex a_0. The full category of bounded-above representations of $\mathrm{Rep}(Q)$ and that of $\mathrm{rep}(Q)$ will be denoted by $\mathrm{Rep}^-(Q)$ and $\mathrm{rep}^-(Q)$, respectively.

Let M be an object in $\mathrm{Rep}(Q)$ and $n \in \mathbb{Z}$. We define an object $M^{\leq n}$ in $\mathrm{Rep}^-(Q)$ as follows: for $x \in Q_0$, we have $M^{\leq n}(x) = M(x)$ if $x \in Q^m$ with $m \leq n$ and $M^{\leq n}(x) = 0$ otherwise; for $\alpha \in Q_1$, we have $M^{\leq n}(\alpha) = M(\alpha)$ if $s(\alpha) \in Q^m$ with $m < n$ and $M^{\leq n}(\alpha) = 0$ otherwise. In a similar manner, we define objects $M^{\geq n}$ and $M^{>n}$ in $\mathrm{Rep}(Q)$.

2.2. Lemma. *Let M, N be objects in* $\mathrm{Rep}(Q)$*, and let be V a k-vector space and n an integer.*

(1) $M^{\leq n} \cong M/M^{>n}$.

(2) $(M \oplus N)^{\leq n} = M^{\leq n} \oplus N^{\leq n}$ *and* $(M \oplus N)^{\geq n} = M^{\geq n} \oplus N^{\geq n}$.

(3) $(M \otimes_k V)^{\leq n} \cong M^{\leq n} \otimes_k V$ *and* $(M \otimes_k V)^{\geq n} \cong M^{\geq n} \otimes_k V$.

Proof. We need only to prove (1), since (2) and (3) are evident. We shall construct an epimorphism $p : M \to M^{\leq n}$ in $\mathrm{Rep}(Q)$ as follows. For $x \in Q_0$, define $p(x) : M(x) \to M^{\leq n}(x)$ by $p(x) = \mathbf{1}_{M(x)}$ if $x \in Q^m$ with $m \leq n$, and otherwise $p(x) = 0$. Let $\alpha : x \to y \in Q_1$ with $x \in Q^m$. Consider the diagram:

$$\begin{tikzcd} M(x) \arrow[r, "M(\alpha)"] \arrow[d, "p(x)"'] & M(y) \arrow[d, "p(y)"] \\ M^{\leq n}(x) \arrow[r, "M^{\leq n}(\alpha)"] & M^{\leq n}(y). \end{tikzcd}$$

If $m < n$, then $M^{\leq n}(\alpha) = M(\alpha)$, $p(x) = \mathbf{1}_{M(x)}$ and $p(y) = \mathbf{1}_{M(y)}$. Otherwise, $M^{\leq n}(\alpha) = 0$ and $p(y) = 0$. Thus the above diagram is commutative in any case. Clearly, the kernel of p is $M^{>n}$. The proof of the lemma is completed.

2.3. Definition. Let M be an object in $\mathrm{Rep}^-(Q)$, and let n be an integer. We say that M is *n-truncated injective* if $M^{\leq n} \cong \oplus_{x \in Q^n} I_x \otimes_k V_x$ with V_x a k-vector space, or equivalently, $M^{\leq n}$ is injective with $\mathrm{soc}(M^{\leq n}) = \oplus_{x \in Q^n} M(x)$. Moreover, M is called *truncated injective* if M is n-truncated injective for some integer n.

Note that a finite dimensional k-representation of Q is truncated injective.

2.4. Lemma. *Let M be object in* $\mathrm{Rep}^-(Q)$*. If M is n-truncated injective for some integer n, then M is m-truncated injective for every $m \leq n$.*

Proof. Assume that M is n-truncated injective and m is an integer with $m < n$. Observe that $M^{\leq m} = (M^{\leq n})^{\leq m}$. By Lemma 2.2(2)(3), we may assume $M^{\leq n} = I_a$ for some $a \in Q^n$. Assume that $S = \{b_1, ..., b_r\}$ is the set of vertices in Q^m which are predecessors of a in Q. For each $1 \leq i \leq r$, let V_i be the

k-vector space having as a basis the set of paths from b_i to a. We now consider the morphism $\phi_i : I_{b_i} \otimes_k V_i \to (I_a)^{\leq m}$ given by $\phi_i(c) = 0$ in case $I_{b_i}(c) = 0$; and otherwise, $\phi_i(c)(\gamma \otimes \rho) = \gamma\rho$, for $\gamma \in I_{b_i}(c)$ and $\rho \in V_i$. Then we have a morphism $\phi = (\phi_1, ..., \phi_r) : \oplus_{i=1}^{r} I_{b_i} \otimes_k V_i \to (I_a)^{\leq m}$. Clearly ϕ is a monomorphism, and it is an epimorphism since every path from a vertex c such that $(I_{b_i})^{\leq m}(c) \neq 0$ to the vertex a is of the form $\gamma\rho$ with ρ a path from some b_i to a. The proof of the proposition is completed.

It follows from Lemmas 1.2(4) and 2.4 that the notion of a truncated injective representation does not depend on the choice of the vertex a_0 for grading Q_0.

2.5. Proposition. *Let M, N be objects in $\mathrm{Rep}^-(Q)$ which are n-truncated injective for some integer n.*

(1) *$M \cong N$ if and only if $M^{\geq n} \cong N^{\geq n}$.*

(2) *$M = 0$ if and only if $M^{\geq n} = 0$.*

(3) *M is indecomposable if and only if $M^{\geq n}$ is indecomposable.*

Proof. Let $\phi : M^{\geq n} \to N^{\geq n}$ be an isomorphism in $\mathrm{Rep}^-(Q)$. In particular, $\phi^n = \oplus_{x \in Q^n} \phi(x) : \oplus_{x \in Q^n} M^{\geq n}(x) \to \oplus_{x \in Q^n} N^{\geq n}(x)$ is an isomorphism from $\mathrm{soc}(M^{\leq n})$ to $\mathrm{soc}(N^{\leq n})$. Since $M^{\leq n}$ and $N^{\leq n}$ are both injective, ϕ^n extends to an isomorphism $\psi : M^{\leq n} \to N^{\leq n}$ in $\mathrm{Rep}^-(Q)$. For each $x \in Q_0$, we define a k-linear map $\zeta(x) : M(x) \to N(x)$ by $\zeta(x) = \phi(x)$ if $x \in Q^m$ with $m \geq n$ and $\zeta(x) = \psi(x)$ otherwise. Since the arrows $x \to y$ in Q are such that $(x, y) \in Q^t \times Q^{t+1}$ for some t, it is easy to verify that $\zeta = \{\zeta(x) \mid x \in Q_0\}$ is an isomorphism in $\mathrm{Rep}^-(Q)$ from M onto N. This establishes (1), and consequently (2) holds.

Assume now that $M^{\geq n}$ is indecomposable. Let $M = M_1 \oplus M_2$. Then M_1, M_2 are n-truncated injective and $M^{\geq n} = M_1^{\geq n} \oplus M_2^{\geq n}$. Thus $M_1^{\geq n} = 0$ or $M_2^{\geq n} = 0$. By (2), we get $M_1 = 0$ or $M_2 = 0$. That is, M is indecomposable. Finally suppose that $M^{\geq n}$ is not indecomposable. If $M^{\geq n} = 0$, then by (2), $M = 0$. Otherwise, $M^{\geq n} = U \oplus V$ with U, V nonzero objects in $\mathrm{Rep}^b(Q)$. Then $\mathrm{soc}(M^{\leq n}) = \oplus_{x \in Q^n} M(x) = \oplus_{x \in Q^n} (U(x) \oplus V(x))$. Since M is n-truncated injective, we may write $M^{\leq n} = I \oplus J$, where I, J are injective objects in $\mathrm{Rep}^-(Q)$ such that $\mathrm{soc}\, I = \oplus_{x \in Q^n} U(x)$ and $\mathrm{soc}\, J = \oplus_{x \in Q^n} V(x)$. We construct a nonzero object $\widetilde{U}$ in $\mathrm{Rep}^-(Q)$ by defining, $\widetilde{U}(x) = I(x)$ if $x \in Q^m$ with $m \leq n$, and otherwise $\widetilde{U}(x) = U(x)$; and $\widetilde{U}(\alpha) = I(\alpha)$ if $s(\alpha) \in Q^m$ with $m < n$, and otherwise $\widetilde{U}(\alpha) = U(\alpha)$. Then $\widetilde{U}^{\leq n} = I$, $\widetilde{U}^{\geq n} = U$. Similarly, we obtain a nonzero object $\widetilde{V}$ in $\mathrm{Rep}^-(Q)$ such that $\widetilde{V}^{\leq n} = J$ and $\widetilde{V}^{\geq n} = V$. Now $\widetilde{U} \oplus \widetilde{V}$ is an object in $\mathrm{Rep}^-(Q)$ such that $M^{\geq n} \cong (\widetilde{U} \oplus \widetilde{V})^{\geq n}$. By (1), $M \cong \widetilde{U} \oplus \widetilde{V}$. This completes the proof of the proposition.

3. The bounded derived category

Recall that a *k-category* is a category in which the morphism sets are k-vector spaces and the composition of morphisms are k-bilinear. Let $\mathfrak{A}$ be an additive k-category which is a full subcategory of an abelian k-category $\mathfrak{B}$. A complex $(X^\bullet, d_X^\bullet)$, or simply $X^\bullet$, in $\mathfrak{A}$ is a double infinite chain

$$\cdots \to X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \to \cdots, \quad n \in \mathbb{Z}$$

of morphisms between objects in $\mathfrak{A}$ such that $d_X^n d_X^{n+1} = 0$ for every integer n, where X^n is called the *component of* $X^\bullet$ *of degree* n, and d_X^n, the *differential* of degree n. Such a complex is called *bounded-above* if $X^n = 0$ for all but finitely many positive integers n; *bounded* if $X^n = 0$ for all but finitely many integers n; and a *stalk complex concentrated in degree* s if $X^n = 0$ for any $n \neq s$. The *n-th cohomology* of $X^\bullet$ is the object $\mathrm{H}^n(X^\bullet) = \mathrm{Ker}(d_X^n)/\mathrm{Im}(d_X^{n-1})$ in $\mathfrak{B}$. One says that $X^\bullet$ has *bounded cohomology* if $\mathrm{H}^n(X^\bullet) = 0$ for all but finitely many integers n. A morphism of complexes $\phi^\bullet : X^\bullet \to Y^\bullet$ consists of a family of morphisms $\phi^n : X^n \to Y^n$ in $\mathfrak{A}$ such that $d_X^n \phi^{n+1} = \phi^n d_Y^n$ for all n. Such a morphism is a called a *quasi-isomorphism* if ϕ^n induces an isomorphism $H^n(X^\bullet) \to H^n(Y^\bullet)$ for each n, and *null-homotopic* if there exist morphisms $h^n : X^n \to Y^{n-1}$ in $\mathfrak{A}$ such that $\phi^n = d_X^n h^{n+1} + h^n d_Y^n$ for all $n \in \mathbb{Z}$. The complexes in $\mathfrak{A}$ form an additive k-category denoted as $C(\mathfrak{A})$. For $X^\bullet \in C(\mathfrak{A})$ and $s \in \mathbb{Z}$, the *shift of* $X^\bullet$ *by* s, written as $X^\bullet[s]$, is the complex whose component and differential of degree n are X^{n+s} and $(-1)^s d_X^{n+s}$, respectively, for any $n \in \mathbb{Z}$. The *homotopy category* $K(\mathfrak{A})$ of $C(\mathfrak{A})$ is its quotient category modulo the ideal of null-homotopic morphisms. This is a triangulated k-category whose translation functor is the shift by 1 and whose exact triangles are induced from the mapping cones. Now the *derived category* $D(\mathfrak{A})$ of $\mathfrak{A}$ is the localization of $K(\mathfrak{A})$ with respect to the quasi-isomorphisms, which is also a triangulated k-category with exact triangles induced from those of $K(\mathfrak{A})$. Moreover, the full subcategories of bounded-above complexes of $C(\mathfrak{A})$, $K(\mathfrak{A})$, and $D(\mathfrak{A})$ will be denoted by $C^-(\mathfrak{A})$, $K^-(\mathfrak{A})$, and $D^-(\mathfrak{A})$, respectively; and those of bounded-above complexes with bounded cohomology will be denoted by $C^{-,b}(\mathfrak{A})$, $K^{-,b}(\mathfrak{A})$, and $D^{-,b}(\mathfrak{A})$, respectively; and those of bounded complexes will be denoted by $C^b(\mathfrak{A})$, $K^b(\mathfrak{A})$, and $D^b(\mathfrak{A})$, respectively. Note that $K^{-,b}(\mathfrak{A})$, $D^{-,b}(\mathfrak{A})$, $K^b(\mathfrak{A})$, and $D^b(\mathfrak{A})$ are all triangulated k-categories. Finally, if one identifies an object in $\mathfrak{A}$ as a stalk complexe concentrated in degree 0, then $\mathfrak{A}$ becomes a full subcategory of each of $C^b(\mathfrak{A})$, $K^b(\mathfrak{A})$ and $D^b(\mathfrak{A})$. We refer to [23] for more details on these notions.

Now let A be a finite-dimensional k-algebra. The k-category of all left A-modules and that of finitely generated ones will be denoted by A-Mod and A-mod, respectively. Moreover, the full subcategories of projective modules of these categories are denoted by A-Proj and A-proj, respectively. Our main interest lies in the derived category $D^b(A)$ of C^b(A-mod), called the *bounded derived category* of A. As usual, we replace $D^b(A)$ by a more accessible category. Indeed, sending a bounded complex in A-mod to its projective resolution yields an equivalence of triangulated categories from $D^b(A)$ to $K^{-,b}(A\text{-proj})$. The quasi-inverse of this equivalence is written as $E : K^{-,b}(A\text{-proj}) \to D^b(A)$. Furthermore, we shall pass from $K^{-,b}(A\text{-proj})$ to another even better behaved category. For this purpose, we need some more terminology. A morphism in A-Mod is called *radical* if its image is contained in the radical of its co-domain; and a complex in A-Mod is called *radical* if the differentials are all radical morphisms. For a full subcategory $\mathfrak{A}$ of A-Mod, we denote by $RC^{-,b}(\mathfrak{A})$ the full subcategory of $C^{-,b}(\mathfrak{A})$ of radical complexes, and consider the canonical projection functor $G : RC^{-,b}(\mathfrak{A}) \to K^{-,b}(\mathfrak{A})$, which acts identically on the objects and sends a morphism to its homotopy class. For collecting the properties of this functor, we recall that a morphism $f : X \to Y$ in an additive category is *left almost split* if f is not a section and every morphism $g : X \to Z$ which is not a section factors through f. Dually, one has the notion of a *right almost morphism.*

3.1. PROPOSITION. *Let A be a finite dimensional k-algebra, and consider the projection functor $G : RC^{-,b}(A\text{-Proj}) \to K^{-,b}(A\text{-Proj})$.*

(1) *A morphism $\phi^\bullet$ in $RC^{-,b}(A\text{-Proj})$ is a section (respectively, retraction) if and only if $G(\phi^\bullet)$ is a section (respectively, retraction) in $K^{-,b}(A\text{-Proj})$.*

(2) *If $\phi^\bullet$ is a left (respectively, right) almost split morphism in $RC^{-,b}(A\text{-proj})$, then $G(\phi^\bullet)$ is let (respectively, right) almost split in $K^{-,b}(A\text{-proj})$.*

(3) *G is dense and preserves indecomposability and isomorphism classes.*

Proof. Let $\phi^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $RC^{-,b}(A\text{-Proj})$. If $\phi^\bullet$ is a section, then $G(\phi^\bullet)$ is clearly a section. Assume now that $G(\phi^\bullet)$ is a section. Let $\psi^\bullet : Y^\bullet \to X^\bullet$ be a morphism in $RC^b(A\text{-Proj})$ such that $G(\phi^\bullet)G(\psi^\bullet) = \mathbf{1}_{G(X^\bullet)}$, that is, $\mathbf{1}_{X^\bullet} - \phi^\bullet\psi^\bullet$ is null-homotopic. In particular, $\mathbf{1}_{X^n} - \phi^n\zeta^n$ are all radical morphisms since $X^\bullet, Y^\bullet$ are radical complexes. Thus $(\mathbf{1}_{X^\bullet} - \phi^\bullet\psi^\bullet)^s = 0$, where s is the nilpotency of rad$A$. As a consequence, $\phi^\bullet\psi^\bullet$ is an automorphism of $X^\bullet$, and hence $\phi^\bullet$ is a section. Assume now that $\phi^\bullet$ is left almost split. Then $G(\phi^\bullet)$ is not a section. Let $\psi^\bullet : X^\bullet \to Z^\bullet$ be a morphism in $RC^b(A\text{-Proj})$ such that $G(\psi^\bullet)$ is not a section. Then $\psi^\bullet : X^\bullet \to Z^\bullet$ is not a section. Thus $\psi^\bullet$ factors through $\phi^\bullet$, and hence $G(\psi^\bullet)$ factors through $G(\phi^\bullet)$. That is, $G(\phi^\bullet)$ is left almost split. This proves (1) and (2).

Since G is full, we deduce immediately from (1) that G preserves isomorphism classes. Let $X^\bullet$ be an object in $RC^{-,b}(A\text{-Proj})$ which is indecomposable in $K^{-,b}(A\text{-Proj})$. Assume that $X^\bullet = Y^\bullet \oplus Z^\bullet$ in $RC^{-,b}(A\text{-Proj})$. Since $X^\bullet$ is indecomposable in $K^{-,b}(A\text{-Proj})$, we may assume that $\mathbf{1}_{Y^\bullet}$ is null-homotopic. Since $Y^\bullet$ is radical, we get $\mathbf{1}^s_{Y^\bullet} = 0$, where s is the nilpotency of rad$A$. Hence $\mathbf{1}_{Y^\bullet} = 0$, that is, $Y^\bullet = 0$. This shows that G preserves indecomposability. Finally let $(X^\bullet, d^\bullet_X)$ be an object in $K^{-,b}(A\text{-Proj})$. We may assume that $\mathrm{H}^n(X^\bullet) = 0$ for all $n \le 0$. Suppose that $X^\bullet$ is not radical. Since $X^\bullet$ is bounded-above, there exists a maximal s such that d_X^{s-1} is not radical. We may assume that d_X^{s-1} is of the following form:

$$d_X^{s-1} = \begin{pmatrix} \mathbf{1}_M & 0 \\ h & g^{s-1} \end{pmatrix} : X^{s-1} = M \oplus N^{s-1} \to M \oplus N^s = X^s,$$

where g^{s-1} is radical. Since d_X^s is radical with $d_X^{s-1}d_X^s = 0$, we have $d_X^s = \binom{0}{g^s}$, where g^s is radical such that $g^{s-1}g^s = 0$. Writing $d_X^{s-2} = (f^{s-2}, g^{s-2})$, we get

$f^{s-2}+g^{s-2}h=0$ and $g^{s-2}g^{s-1}=0$. Suppose that $s>0$. Define $Y^n=N^n$ if $s-1\le n\le s$ and otherwise $Y^n=X^n$. Moreover, let $d_Y^n=g^n$ if $s-2\le n\le s$ and otherwise $d_Y^n=d_X^n$. Then $(Y^\bullet, d_Y^\bullet)$ is a complex in $C^{-,b}(A\text{-proj})$ such that $\mathrm{H}^n(Y^\bullet)=0$ for $n\le 0$, and d_Y^n is radical for $n\ge s-1$. Let $\phi^n=\binom{0}{\mathbf{1}_{N^n}}$ if $s-1\le n\le s$ and $\phi^n=\mathbf{1}_{X^n}$ otherwise; and let $\psi^{s-1}=(-h,\mathbf{1}_{N^{s-1}})$, $\psi^s=(0,\mathbf{1}_{N^s})$, and $\psi^n=\mathbf{1}_{X^n}$ for $n\ne s-1,s$. Then $\phi^\bullet=\{\phi^n\mid n\in\mathbb{Z}\}$ and $\psi^\bullet=\{\psi^n\mid n\in\mathbb{Z}\}$ are morphisms in $C^{-,b}(A\text{-proj})$ such that $\psi^\bullet\phi^\bullet=\mathbf{1}_{Y^\bullet}$ and $\phi^\bullet\psi^\bullet$ is homotopic to $\mathbf{1}_{X^\bullet}$. Thus $X^\bullet\cong Y^\bullet$ in $K^{-,b}(A\text{-Proj})$. By induction, we may assume that $\mathrm{H}^n(X^\bullet)=0$ for $n\le 0$ and d_X^n is radical for $n\ge 0$. Write $d_X^0=pj$, where $p: X^0\to L$ is an epimorphism and $j: L\to X^1$ is an monomorphism. Let

$$\cdots\longrightarrow P^n\xrightarrow{d^n}P^{n+1}\longrightarrow\cdots\longrightarrow P^{-1}\xrightarrow{d^{-1}}P^0\xrightarrow{d^0}L\to 0$$

be a minimal projective resolution of L in A-Mod. Define $Z^n=P^n$ for $n\le 0$, and $Z^n=X^n$ for $n>0$. Moreover, let $d_Z^n=d^n$ for $n<0$, $d_Z^n=d_X^n$ for $n>0$, and $d_Z^0=d^0j$. Then $(Z^\bullet, d_Z^\bullet)\in RC^{-,b}(A\text{-Proj})$ such that $X^\bullet\cong Z^\bullet$ in $K^{-,b}(A\text{-Proj})$. The proof of the proposition is completed.

For the rest of this section, assume that A is connected and elementary with $\mathrm{rad}^2(A)=0$. Then $A\cong kQ/I$, where Q is the ordinary quiver of A which is connected and finite, and I is the ideal in kQ generated by the paths of length two. For simplifying the notation, we shall assume throughout that $A=kQ/I$. For a vertex a in Q, we write $P[a]=Ae_a$ with $e_a=\varepsilon_a+I$, where ε_a is the trivial path in Q at a, and $S[a]=P[a]/\mathrm{rad}P[a]$. For an arrow $\alpha: a\to b$ in Q, let $P[\alpha]: P[a]\to P[b]$ denote the right multiplication by $\bar\alpha=\alpha+I$, and for a trivial path ε_a, let $P[\varepsilon_a]=\mathbf{1}_{P[a]}$. Fix a minimal gradable covering $\pi:\widetilde{Q}\to Q$ of Q. Since Q is finite, $\widetilde{Q}$ is locally finite. Choose a vertex $\tilde{x}$ in $\widetilde{Q}$ and write $\widetilde{Q}^n=\widetilde{Q}^n(\tilde{x})$ for all $n\in\mathbb{Z}$. It follows from Lemma 1.5 that the $\widetilde{Q}^n$ are all finite and the arrows in $\widetilde{Q}$ are of the form $x\to y$ with $x\in\widetilde{Q}^n$ and $y\in\widetilde{Q}^{n+1}$ for some n. As a consequence, $\widetilde{Q}$ is interval-finite. We shall write $u^\pi=\pi(u)$ for $u\in\widetilde{Q}_0\cup\widetilde{Q}_1$. All tensor products in this section are over the ground field k.

3.2. Lemma. *Let U,V be k-vector spaces. If a,b are vertices in Q, then every A-linear map $\phi: P[a]\otimes U\to P[b]\otimes V$ can be uniquely written as*

$$\phi=\sum\nolimits_{\rho\in Q_{\le 1}(a,b)}P[\rho]\otimes f_\rho,\;\; f_\rho\in\mathrm{Hom}_k(U,V).$$

Moreover, ϕ is radical if and only if $\phi=\sum_{\alpha\in Q_1(a,b)}P[\alpha]\otimes f_\alpha,\; f_\alpha\in\mathrm{Hom}_k(U,V)$.

Proof. Every A-linear map $\phi: P[a]\otimes U\to P[b]\otimes V$ is uniquely determined by its restriction to $e_a\otimes U$, which yields a k-linear map $f: e_a\otimes U\to(e_aAe_b)\otimes V$. Conversely every k-linear map $f: e_a\otimes U\to(e_aAe_b)\otimes V$ can be extended in a unique way to an A-linear map $\phi: P[a]\otimes U\to P[b]\otimes V$. Since $\mathrm{rad}^2(A)=0$, we have $(e_aAe_b)\otimes V=\oplus_{\rho\in Q_{\le 1}(a,b)}(k\overline{\rho})\otimes V$. Observe that $\phi=P[\rho]\otimes f_\rho$ for some $f_\rho\in\mathrm{Hom}_k(U,V)$ if and only if $\phi(e_a\otimes U)\subseteq(k\overline{\rho})\otimes V$. Now every k-linear map $f: e_a\otimes U\to(e_aAe_b)\otimes V$ can be uniquely written as a sum $f=\sum_{\rho\in Q_{\le 1}(a,b)}f_\rho$

with $f_\rho(e_a \otimes U) \subseteq (k\overline{\rho}) \otimes V$. Thus every A-linear map $\phi : P[a] \otimes U \to P[b] \otimes V$ can be uniquely written as a sum $\phi = \sum_{\rho \in Q_{\le 1}(a,b)} P[\rho] \otimes f_\rho$ with $f_\rho \in \mathrm{Hom}_k(U,V)$.

Finally, let $\phi = \sum_{\rho \in Q_{\le 1}(a,b)} P[\rho] \otimes f_\rho$ with $f_\rho \in \mathrm{Hom}_k(U,V)$ be an A-linear map. Note that $\mathrm{rad}(P[b] \otimes V) = (\mathrm{rad}P[b]) \otimes V$. If $\phi = \sum_{\alpha \in Q_1(a,b)} P[\alpha] \otimes f_\alpha$, then ϕ is clearly radical. Otherwise, $a = b$ and $f_{\varepsilon_a}(u) \neq 0$ for some $u \in U$. Now $\phi(e_a \otimes u) = e_a \otimes f_{\varepsilon_a}(u) \notin \mathrm{rad}(P[b] \otimes V)$. That is, ϕ is not radical. The proof of the lemma is completed.

Let M be a k-representation of $\widetilde{Q}$. We shall construct a radical complex $(F(M)^\bullet, d^\bullet_{F(M)})$ in A-Proj. For $n \in \mathbb{Z}$, let $F(M)^n = \oplus_{x \in \widetilde{Q}^n} P[x^\pi] \otimes M(x)$ and $d^n_{F(M)} : F(M)^n \to F(M)^{n+1}$ be the A-linear map given by the matrix $(d^n_{F(M)}(x,y))_{(x,y) \in \widetilde{Q}^n \times \widetilde{Q}^{n+1}}$, where

$$d^n_{F(M)}(x,y) = \sum\nolimits_{\alpha \in \widetilde{Q}(x,y)} P[\alpha^\pi] \otimes M(\alpha) : P[x^\pi] \otimes M(x) \to P[y^\pi] \otimes M(y).$$

Since $\mathrm{rad}^2(A) = 0$, $(F(M)^\bullet, d^\bullet_{F(M)})$ is indeed a radical complex. Let $f : M \to N$ be a morphism in $\mathrm{Rep}(\widetilde{Q})$. For each integer n, we define

$$F(f)^n = \oplus_{x \in \widetilde{Q}^n} \mathbf{1}_{P[x^\pi]} \otimes f(x) : F(M)^n \to F(N)^n.$$

It is easy to verify that $F(f)^\bullet = \{F(f)^n \mid n \in \mathbb{Z}\}$ is a morphism in $RC(A\text{-Proj})$ from $F(M)^\bullet$ to $F(N)^\bullet$. This yields a k-linear functor

$$F : \mathrm{Rep}(\widetilde{Q}) \to RC(A\text{-Proj}),$$

which is said to be *induced from the covering* $\pi : \widetilde{Q} \to Q$.

We need some more notation. Let $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$. Denote by $r(\sigma)$ the integer such that $\sigma(\widetilde{Q}^0) = \widetilde{Q}^{r(\sigma)}$. It follows from Lemma 1.5(4) that $\sigma(\widetilde{Q}^n) = \widetilde{Q}^{n+r(\sigma)}$, for every integer n. If $M \in \mathrm{Rep}(\widetilde{Q})$, we define the *$\sigma$-translate* M^σ of M by $M^\sigma(x) = M(\sigma(x))$ and $M^\sigma(\alpha) = (-1)^{r(\sigma)} M(\sigma(\alpha))$, for $x \in \widetilde{Q}_0$ and $\alpha \in \widetilde{Q}_1$.

3.3. Lemma. *The functor $F : \mathrm{Rep}(\widetilde{Q}) \to RC(A\text{-Proj})$ is faithful and exact. Moreover, if $M \in \mathrm{Rep}(\widetilde{Q})$ and $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$, then $F(M^\sigma)^\bullet = F(M)^\bullet[r(\sigma)]$.*

Proof. The first part follows immediately from the fact that the tensor product involved in the definition of F is over the field k. Let $M \in \mathrm{Rep}(\widetilde{Q})$ and $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$. For each $n \in \mathbb{Z}$, we have

$$F(M^\sigma)^n = \oplus_{x \in \widetilde{Q}^n} P[x^\pi] \otimes M^\sigma(x) = \oplus_{x \in \widetilde{Q}^n} P[\sigma(x)^\pi] \otimes M(\sigma(x)) = F(M)^{n+r(\sigma)},$$

where the last equality holds since $\sigma(\widetilde{Q}^n) = \widetilde{Q}^{n+r(\sigma)}$. Moreover, for each pair $(x,y) \in \widetilde{Q}^n \times \widetilde{Q}^{n+1}$,

$$\begin{aligned} d^n_{F(M^\sigma)}(x,y) &= \sum\nolimits_{\alpha \in \widetilde{Q}_1(x,y)} P[\alpha^\pi] \otimes M^\sigma(\alpha) \\ &= (-1)^{r(\sigma)} \sum\nolimits_{\alpha \in \widetilde{Q}_1(x,y)} P[\sigma(\alpha)^\pi] \otimes M(\sigma(\alpha)) \\ &= (-1)^{r(\sigma)} d^{n+r(\sigma)}_{F(M)}(\sigma(x), \sigma(y)), \end{aligned}$$

where the last equality follows from the equality $\sigma(\widetilde{Q}_1(x,y)) = \widetilde{Q}_1(\sigma(x),\sigma(y))$. Therefore, $d^n_{F(M^\sigma)} = (-1)^{r(\sigma)} d^{n+r(\sigma)}_{F(M)}$. This shows that $F(M^\sigma)^\bullet = F(M)^\bullet[r(\sigma)]$. The proof of the lemma is completed.

For $M \in \mathrm{Rep}(\widetilde{Q})$, it is clear that $F(M)^\bullet \in RC^-(A\text{-Proj})$ if and only if $M \in \mathrm{Rep}^-(\widetilde{Q})$.

3.4. Lemma. *Every indecomposable object in* $RC^-(A\text{-Proj})$ *is isomorphic to a shift of some* $F(M)^\bullet$ *with* M *an indecomposable object in* $\mathrm{Rep}^-(\widetilde{Q})$.

Proof. Assume that $(X^\bullet, d^\bullet_X)$ is an indecomposable object in $RC^-(A\text{-Proj})$. Let Λ be the set of integers n for which $X^n \neq 0$. Since $X^\bullet$ is indecomposable and bounded-above, Λ is interval-closed with a maximal element r. For $n \in \Lambda$, write

$$X^n \cong \oplus_{a\in S(n)} P[a] \otimes V(n,a),$$

where $S(n)$ is a subset of Q_0 and $V(n,a)$ is a non-zero k-vector space. For $n \in \Lambda\backslash\{r\}$, write $d^n_X = (d^n_X(a,b))_{(a,b)\in S(n)\times S(n+1)}$ with $d^n_X(a,b)$ some A-linear map from $P[a]\otimes V(n,a)$ to $P[b]\otimes V(n+1,b))$. By Lemma 3.2,

$$d^n_X(a,b) = \sum\nolimits_{\alpha\in Q_1(a,b)} P[\alpha]\otimes f(n,\alpha)$$

with $f(n,\alpha) \in \mathrm{Hom}_k(V(n,a), V(n+1,b))$.

Choose arbitrarily a vertex a_r in $S(r)$ and let x_r be some vertex in $\widetilde{Q}$ with $x_r^\pi = a_r$. Shifting $X^\bullet$ if necessary, we may assume that $x_r \in \widetilde{Q}^r$. Let $n \in \Lambda$. For each $a \in S(n)$, we claim that there exists an unique x in $\widetilde{Q}^n$ such that $x^\pi = a$. Indeed, the uniqueness follows from Theorem 1.3. For the existence, we may assume that $a \neq a_r$. Since $X^\bullet$ is indecomposable, there exist integers $n = n_0, n_1, \ldots, n_s = r$ in Λ with $s > 0$ and $n_i = n_{i+1} \pm 1$, and vertices $a = b_0, b_1, \ldots, b_s = a_r$ in Q with $b_i \in S(n_i)$ such that $d^{n_i}_X(b_i, b_{i+1}) \neq 0$ if $n_i = n_{i+1} - 1$ or $d^{n_{i+1}}_X(b_{i+1}, b_i) \neq 0$ if $n_i = n_{i+1}+1$. This gives rise to a walk $w = \alpha_1^{n_1-n_0}\alpha_2^{n_2-n_1}\cdots\alpha_s^{n_s-n_{s-1}}$ in Q from a to a_r. Let $\widetilde{w}$ be the walk in $\widetilde{Q}$ with $e(\widetilde{w}) = x_r$ such that $\pi(\widetilde{w}) = w$. Letting $x = s(\widetilde{w})$, we get $x^\pi = a$. Since $\partial(\widetilde{w}) = \partial(w) = \sum_{0\le i<s}(n_{i+1}-n_i) = r-n$, we have $x \in \widetilde{Q}^n$ by Lemma 1.2(2). This establishes our claim. Set $\widetilde{S}(n) = \{x \in \widetilde{Q}^n \mid x^\pi \in S(n)\}$.

Next we define an object $M \in \mathrm{Rep}^-(\widetilde{Q})$ as follows. For each vertex x in $\widetilde{Q}$, define $M(x) = V(n, x^\pi)$ if $x \in \widetilde{S}(n)$ with $n \in \Lambda$; and $M(x) = 0$ otherwise. For each arrow $\beta : x \to y$ in $\widetilde{Q}$, define $M(\beta) = f(n, \beta^\pi)$ if $(x,y) \in \widetilde{S}(n)\times\widetilde{S}(n+1)$ with $n \in \Lambda\backslash\{r\}$; and $M(\beta) = 0$ otherwise.

It remains to show that $F(M)^\bullet = X^\bullet$. Let n be any integer. If $n \notin \Lambda$, then it is evident that $F(M)^n = 0 = X^n$. If $n \in \Lambda$, then it follows from the definition of M and our previous claim that

$$F(M)^n = \oplus_{x\in\widetilde{S}(n)} P[x^\pi]\otimes V(n,x^\pi) = \oplus_{a\in S(n)} P[a]\otimes V(n,a) = X^n.$$

Moreover, if $n = r$ or $n \notin \Lambda$, then $d^n_{F(M)} = 0 = d^n_X$. Assume that $n \in \Lambda$ with $n < r$. By the definition of M, $d^n_{F(M)} = (d^n_{F(M)}(x,y))_{(x,y)\in\widetilde{S}(n)\times\widetilde{S}(n+1)}$, where $d^n_{F(M)}(x,y) = \sum_{\beta\in\widetilde{Q}_1(x,y)} P[\beta^\pi]\otimes f(n,\beta^\pi)$. Now for each pair (a,b) in $S(n)\times S(n+1)$, there exists an unique pair (x,y) in $\widetilde{S}(n)\times\widetilde{S}(n+1)$ with $(x^\pi, y^\pi) = (a,b)$. By Lemma 1.5(3), we have

$$\sum\nolimits_{\beta\in\widetilde{Q}_1(x,y)} P[\beta^\pi]\otimes f(n,\beta^\pi) = \sum\nolimits_{\alpha\in Q_1(a,b)} P[\alpha]\otimes f(n,\alpha),$$

that is, $d^n_{F(M)}(x,y) = d^n_X(a,b)$. As a consequence, $d^n_{F(M)} = d^n_X$. This shows that $F(M)^\bullet = X^\bullet$. The proof of the lemma is completed.

3.5. Lemma. *Let M be an object in $\mathrm{Rep}^-(\widetilde{Q})$, and let $\phi^\bullet : F(M)^\bullet \to X^\bullet$ be a nonzero morphism in $RC^-(A\text{-Proj})$ with $X^\bullet$ indecomposable. If $\phi^\bullet$ is non-radical, then $X^\bullet \cong F(N)^\bullet$, and otherwise, $X^\bullet \cong F(N)^\bullet[1]$, where N is some object in $\mathrm{Rep}^-(\widetilde{Q})$.*

Proof. By Lemma 3.4, we may assume that $X^\bullet = F(L)^\bullet[s]$ with $L \in \mathrm{Rep}^-(\widetilde{Q})$ and $s \in \mathbb{Z}$. For each integer n, write $\phi^n = (\phi^n(x,y))_{(x,y)\in\widetilde{Q}^n\times\widetilde{Q}^{n+s}}$, where $\phi^n(x,y) : P[x^\pi]\otimes M(x) \to P[y^\pi]\otimes L(y)$ is an A-linear map. Assume first that ϕ is radical. Let m be such that $\phi^m \neq 0$. Then there exists $(x,y)\in\widetilde{Q}^m\times\widetilde{Q}^{m+s}$ such that $\phi^m(x,y)\neq 0$. By Lemma 3.2, Q has an arrow $x^\pi \to y^\pi$, which is lifted to an arrow $x\to z$ in $\widetilde{Q}$. Then $z\in\widetilde{Q}^{m+1}$ such that $z^\pi = y^\pi$, that is $\pi(\widetilde{Q})^{m+s}\cap\pi(\widetilde{Q}^{m+1})\neq\emptyset$. By Lemmas 1.7(4), there exists some $\sigma\in\mathrm{Aut}_\pi(\widetilde{Q})$ such that $\sigma(\widetilde{Q}^{m+s}) = \widetilde{Q}^{m+1}$. By Lemma 1.5(4), $\sigma(\widetilde{Q}^0) = \widetilde{Q}^{s-1}$, that is, $r(\sigma) = s-1$. By Lemma 3.3, $F(L^\sigma)^\bullet = F(L)^\bullet[s-1]$. Thus $X^\bullet = F(L)^\bullet[s] = F(L^\sigma)^\bullet[1]$. If ϕ is not radical, then there exists $(a_0,b_0)\in\widetilde{Q}^m\times\widetilde{Q}^{m+s}$ such that $\phi^m(x_0,y_0)$ is not radical for some $m\in\mathbb{Z}$. Therefore, $\pi(\widetilde{Q}^{m+s})\cap\pi(\widetilde{Q}^m)\neq\emptyset$. As argued previously, $F(L)^\bullet[s] = F(L^\theta)^\bullet$ for some $\theta\in\mathrm{Aut}_\pi(\widetilde{Q})$. This completes the proof of the lemma.

3.6. Lemma. *Let $\phi^\bullet : F(M)^\bullet \to F(N)^\bullet$ be a morphism in $RC^-(A\text{-Proj})$ with M, N some objects in $\mathrm{Rep}^-(\widetilde{Q})$. Then there exists a morphism $f : M\to N$ in $\mathrm{Rep}^-(\widetilde{Q})$ such that $\phi^\bullet = F(f)^\bullet + \psi^\bullet$ with $\psi^\bullet$ a radical morphism. In this case, $\phi^\bullet$ is a section (retraction) if and only if f is a section (retraction). Moreover, if $M = N$ and $\phi^\bullet$ is an idempotent, then f is an idempotent.*

Proof. For each integer n, write $\phi^n = (\phi^n(x,y))_{(x,\,y)\in\widetilde{Q}^n\times\widetilde{Q}^n}$, where $\phi^n(x,y)$ is an A-linear map from $P[x^\pi]\otimes M(x)$ to $P[y^\pi]\otimes N(y)$. For $(x,y)\in\widetilde{Q}^n\times\widetilde{Q}^n$, set $\delta_{xy} = \mathbf{1}_{P[x^\pi]}$ if $x = y$, and $\delta_{xy} = 0$ otherwise. Note that $x^\pi = y^\pi$ only if $x = y$. We deduce from Lemma 3.2 that $\phi^n(x,y) = \delta_{xy}\otimes f_{xy} + \sum_{\alpha\in Q_1(x^\pi,\,y^\pi)} P[\alpha]\otimes f_\alpha$, where $f_{xy},\ f_\alpha \in \mathrm{Hom}_k(M(x), N(y))$. Put $\psi^n(x,y) = \sum_{\alpha\in Q_1(x^\pi,y^\pi)} P[\alpha^\pi]\otimes f_\alpha$. Then $\psi^n = (\psi^n(x,y))_{(x,\,y)\in\widetilde{Q}^n\times\widetilde{Q}^n} : F(M)^n \to F(N)^n$ is a radical A-linear map. Since $\mathrm{rad}^2(A)$ vanishes, $\psi^\bullet = \{\psi^n \mid n\in\mathbb{Z}\}$ is a radical morphism in $RC^-(A\text{-Proj})$ from $F(M)^\bullet$ to $F(N)^\bullet$. Setting $\eta^n = \oplus_{x\in\widetilde{Q}^n}\mathbf{1}_{P[x^\pi]}\otimes f_{xx}$, we get a morphism $\eta^\bullet = \{\eta^n \mid n\in\mathbb{Z}\}$ in $RC^{-,b}(A\text{-Proj})$ from $F(M)^\bullet$ to $F(N)^\bullet$. It remains to verify that $f = \{f_{xx} \mid x\in\widetilde{Q}_0\}$ is a morphism in $\mathrm{Rep}^-(\widetilde{Q})$ from M to N. Indeed, let $x,y\in\widetilde{Q}_0$ be such that $\widetilde{Q}_1(x,y)$ is non-empty, and assume that $x \in \widetilde{Q}^n$, and hence $y \in \widetilde{Q}^{n+1}$, for some n. We deduce from the equality $d^n_{F(M)}\eta^{n+1} = \eta^n d^n_{F(N)}$ that

$$\sum\nolimits_{\beta \in \widetilde{Q}_1(x,y)} P[\beta^\pi] \otimes M(\beta) f_y = \sum\nolimits_{\beta \in \widetilde{Q}_1(x,y)} P[\beta^\pi] \otimes f_x N(\beta).$$

Since $\pi : \widetilde{Q}_1(x,y) \to Q_1(x^\pi, y^\pi)$ is bijective by Lemma 1.5(3), we deduce from the uniqueness stated in Lemma 3.2 that $M(\beta)f_y = f_x M(\beta)$, for all $\beta \in \widetilde{Q}_1(x,y)$. That is, $f \in \mathrm{Rep}^-(\widetilde{Q})$ such that $\eta^\bullet = F(f)^\bullet$. This establishes the first part of the lemma.

Suppose now that $\phi^\bullet$ is a section. Then $\phi^\bullet\zeta^\bullet = \mathbf{1}_{F(M)^\bullet}$ for some morphism $\zeta^\bullet : F(N)^\bullet \to F(M)^\bullet$ in $RC^-(A\text{-Proj})$. As we have just shown, there exists a morphism $g : N \to M$ in $\mathrm{Rep}^-(\widetilde{Q})$ such that $\zeta^\bullet = F(g)^\bullet + \eta^\bullet$, where $\eta^\bullet$ is radical. This gives rise to $F(\mathbf{1}_M - fg)^\bullet = \mathbf{1}_{F(M)^\bullet} - F(fg)^\bullet = \psi^\bullet\eta^\bullet$. Being radical, $F(\mathbf{1}_M - fg)^\bullet$ is squared zero since $(\mathrm{rad}A)^2 = 0$. Thus $(\mathbf{1}_M - fg)^2 = 0$ since F is faithful. Consequently, fg is an automorphism of M, and hence f is a section. Conversely, assume that there exists a morphism $h : N \to M$ such that $fh = \mathbf{1}_M$. Then $\phi^\bullet F(h)^\bullet = \mathbf{1}_{F(M)^\bullet} + \psi^\bullet F(h)^\bullet$. It follows from $\mathrm{rad}^2(A) = 0$ that $(\mathbf{1}_{F(M)^\bullet} - \phi^\bullet F(h)^\bullet)^2 = 0$. As a consequence, $\phi^\bullet$ is a section. Similarly, one can show that f is a retraction if and only if $\phi^\bullet$ is a retraction.

Finally assume that $M = N$. If $\phi^\bullet$ is an idempotent, then $\phi^n(x,x) = \sum_{y \in \widetilde{Q}^n} \phi^n(x,y)\phi^n(y,x)$ for all $x \in \widetilde{Q}^n$. This yields

$$\mathbf{1} \otimes (f_{xx} - f_{xx}^2) + \sum\nolimits_{\alpha \in Q_1(x^\pi, x^\pi)} P[\alpha] \otimes (f_\alpha - f_\alpha f_{xx} - f_{xx} f_\alpha) = 0,$$

for all $x \in \widetilde{Q}^n$. By the uniqueness stated in Lemma 3.2 we have $f_{xx}^2 = f_{xx}$, and consequently $f^2 = f$. The proof of the lemma is completed.

Next we shall determine the representations M in $\mathrm{Rep}^-(\widetilde{Q})$ such that $F(M)^\bullet$ lies in $RC^{-,b}(A\text{-Proj})$. For this purpose we choose, for each module N in A-Mod, a minimal projective resolution $P_N^\bullet$ of N which is the object in $RC^{-,b}(A\text{-Proj})$ such that $P_N^n = 0$ for $n > 0$, P_N^0 is the projective cover of N, and $\mathrm{H}^n(P_N^\bullet) = 0$ for $n < 0$.

3.7. Lemma. *Let I_x be the indecomposable injective k-representation of $\widetilde{Q}$ associated to a vertex x. If $x \in \widetilde{Q}^n$, then $F(I_x)^\bullet = P_S^\bullet[-n]$, where S is the simple A-module supported by x^π.*

Proof. We need only to consider the case where $x \in \widetilde{Q}^0$. Note that a path p in $\widetilde{Q}$ ending in x is of length t if and only if $s(p) \in \widetilde{Q}^{-t}$. For each $m \le 0$, let $\Theta_m = \{p_{m1}, \ldots, p_{m,s_m}\}$ with $s_m \ge 0$ be the set of paths of length $-m$ ending in x. Write $y_{mi} = s(p_{mi})$ and $b_{mi} = \pi(y_{mi})$, $i = 1, \ldots, s_m$. Then $\oplus_{y \in \widetilde{Q}^m} I_x(y)$ has as a k-basis the set $\{p_{m1}, \ldots, p_{m,s_m}\}$. For each $m < 0$, write $p_{mi} = \alpha_{mi} q_{m+1,i}$ with $\alpha_{mi} \in \widetilde{Q}_1$ and $q_{m+1,i} \in \Theta_{m+1}$, $i = 1, \ldots, s_m$. Then $F(I_x)^m = \oplus_{i=1}^{s_m} P[b_{mi}] \otimes I_x(y_{mi})$ has as a k-basis the set

$$\{\bar{\varepsilon}_{y_{mi}} \otimes p_{mi},\ \bar{\alpha}_{m-1,j} \otimes q_{mj} \mid i = 1, \ldots, s_m; j = 1, \ldots, s_{m-1}\},$$

where $\bar{u} = u + I \in A$. By definition, $d^m_{F(I_x)}$ sends $\bar{\varepsilon}_{y_{mi}} \otimes p_{mi}$ to $\overline{\alpha}_{mi} \otimes q_{m+1,i}$, $i = 1, \ldots, s_m$ and vanishes on $\overline{\alpha}_{m-1,j} \otimes q_{mj}$, $j = 1, \ldots, s_{m-1}$. Therefore, the kernel of $d^m_{F(I_x)}$ has as a k-basis the set $\{\overline{\alpha}_{m-1,j} \otimes q_{mj} \mid j = 1, \ldots, s_{m-1}\}$, which is a k-basis of the image of $d^{m-1}_{F(I_x)}$. Hence, $\mathrm{H}^m(F(I_x)) = 0$ for all $m < 0$. Since $F(I_x)^0 = P[x^\pi]$, we get $F(I_x)^\bullet = P^\bullet_S$. The proof of the lemma is completed.

3.8. Proposition. *An object M in $\mathrm{Rep}^-(\widetilde{Q})$ is truncated injective if and only if $F(M)^\bullet$ has bounded cohomology.*

Proof. First let n be an integer such that $M^{\le n} = \oplus_{i=1}^r (I_{x_i} \otimes V_i)$, where the x_i are vertices in $\widetilde{Q}^n$ and the V_i are k-vector spaces. It follows easily from the definition of F that $F(N \otimes V)^\bullet \cong F(N)^\bullet \otimes V$ for any object N in $\mathrm{Rep}(\widetilde{Q})$ and k-vector space V. Therefore, for $m < n$, we have

$$\mathrm{H}^m(F(M)^\bullet) = \mathrm{H}^m(F(M^{\le n})^\bullet) = \oplus_{i=1}^r \mathrm{H}^m(F(I_{x_i})^\bullet) \otimes V_i,$$

which vanishes by Lemma 3.7. Thus $F(M)^\bullet \in RC^{-,b}(A\text{-Proj})$. Conversely, let n be an integer such that $\mathrm{H}^m(F(M)^\bullet) = 0$ for all $m \le n$. Let S be the image of $d^n_{F(M)}$. Then $F(M^{\le n})^\bullet \cong P^\bullet_S[-n]$. If $S = 0$, then $F(M^{\le n})^\bullet = 0$. Thus $M^{\le n} = 0$, and M is n-truncated injective. Otherwise, S is semi-simple. Hence $S \cong \oplus_{i=1}^s S[y_i^\pi] \otimes U_i$, where $y_1, \ldots, y_s \in \widetilde{Q}^n$ and $U_1, \ldots, U_s$ are some k-vector spaces, and $F(M)^n \cong \oplus_{i=1}^s P[y_i^\pi] \otimes U_i$. It follows from Lemma 3.7 that $F(M^{\le n})^\bullet \cong F(\oplus_{i=1}^s I_{y_i} \otimes U_i)^\bullet$. We deduce easily from the second part of Lemma 3.6 that $M^{\le n} \cong \oplus_{i=1}^s I_{y_i} \otimes U_i$. That is, M is n-truncated injective. The proof of the proposition is completed.

The full subcategory of truncated injective representations of $\mathrm{Rep}^-(\widetilde{Q})$ and that of $\mathrm{rep}^-(\widetilde{Q})$) will be denoted by $\mathrm{Rep}^{-,i}(\widetilde{Q})$ and $\mathrm{rep}^{-,i}(\widetilde{Q})$, respectively. By Proposition 3.8, the functor $F : \mathrm{Rep}(\widetilde{Q}) \to RC(A\text{-Proj})$ induces functors $\mathrm{Rep}^{-,i}(\widetilde{Q}) \to RC^{-,b}(A\text{-Proj})$ and $\mathrm{rep}^{-,i}(\widetilde{Q}) \to RC^{-,b}(A\text{-proj})$ which, by abuse of notation, will be denoted again by F.

3.9. Lemma. *The functor $F : \mathrm{Rep}^{-,i}(\widetilde{Q}) \to RC^{-,b}(A\text{-Proj})$ preserves isomorphism classes and indecomposability. Moreover, F is fully faithful in case the grading period of Q is different from 1.*

Proof. It follows easily from the second part of Lemma 3.6 that F preserves isomorphism classes. Let M be an object in $\mathrm{Rep}^{-,i}(\widetilde{Q})$. If $F(M)^\bullet$ is indecomposable, then M is clearly indecomposable. Suppose now that M is indecomposable. Let $e^\bullet$ be an idempotent endomorphism of $F(M)^\bullet$. By the last part of Lemma 3.6, $e^\bullet = F(f)^\bullet + \psi^\bullet$, where f is an idempotent endomorphism of M and $\psi^\bullet$ a radical morphism. If $f = \mathbf{1}_M$, then $e^\bullet$ is an isomorphism, and hence $e^\bullet = \mathbf{1}_{F(M)^\bullet}$. If $f = 0$, then $e^\bullet$ is radical, and hence nilpotent. Therefore $e^\bullet = 0$. This shows that $F(M)^\bullet$ is indecomposable.

For proving the second part of the lemma, we note that F is always faithful by Lemma 3.3. Suppose that there exists some morphism $\phi^\bullet : F(M)^\bullet \to F(N)^\bullet$ in $RC^{-,b}(A\text{-Proj})$, where $M, N \in \mathrm{Rep}^{-,i}(\widetilde{Q})$, such that $\phi^\bullet \ne F(f)^\bullet$ for any morphism $f : M \to N$ in $\mathrm{Rep}^{-,i}(\widetilde{Q})$. By the first part of Lemma 3.6, there exists a non-zero radical morphism $\eta^\bullet : F(M)^\bullet \to F(N)^\bullet$. As argued in the proof of Lemma 3.5, we see that $\pi(\widetilde{Q}^m) \cap \pi(\widetilde{Q}^{m+1}) \neq \emptyset$ for some integer m. By Proposition 1.7, m and $m+1$ are congruent modulo the grading period of Q. Hence the grading period of Q is 1. This completes the proof of the lemma.

3.10. Lemma. *Let M be a non-zero object in $\mathrm{Rep}^{-,i}(\widetilde{Q})$. Two integers s, t are congruent modulo the grading period of Q if and only if $F(M)^\bullet[s] \cong F(N)^\bullet[t]$ for some $N \in \mathrm{Rep}^{-,i}(\widetilde{Q})$. In this case, $N \cong M^\sigma$ for some $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$.*

Proof. Let r be the grading period of Q. Assume that $s \equiv t \,(\mathrm{mod}\, r)$. By Lemmas 1.7(4) and 1.5(4), there exists $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$ with $r(\sigma) = s - t$. By Lemma 3.3, $F(M^\sigma)^\bullet[t] \cong F(M)^\bullet[s]$. Suppose conversely that there exists an isomorphism $\phi^\bullet : F(M)^\bullet[s] \to F(N)^\bullet[t]$ in $\mathrm{Rep}^{-,i}(\widetilde{Q})$. As argued in the proof of Lemma 3.5, $\pi(\widetilde{Q}^{m+s}) \cap \pi(\widetilde{Q}^{m+t}) \neq \emptyset$ for some integer m. By Proposition 1.7, $s \equiv t \,(\mathrm{mod} r)$ and there exists some $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$ with $r(\sigma) = s - t$. By Lemma 3.3, $F(M^\sigma)^\bullet[t] = F(M)^\bullet[s]$, and thus $F(M^\sigma)^\bullet \cong F(N)^\bullet$. By Lemma 3.9, $N \cong M^\sigma$. The proof of the lemma is completed.

We are now ready to describe the indecomposable objects and some morphism spaces in $D^b(A)$ in terms of those in $\mathrm{rep}^{-,i}(\widetilde{Q})$. We call the composite

$$\mathcal{F} :\ \mathrm{rep}^{-,i}(\widetilde{Q}) \xrightarrow{F} RC^{-,b}(A\text{-proj}) \xrightarrow{G} K^{-,b}(A\text{-proj}) \xrightarrow{E} D^b(A)$$

the *functor induced* from the minimal gradable covering $\pi : \widetilde{Q} \to Q$. Moreover, we denote by $\mathrm{ind}^{-,i}(\widetilde{Q})$ a complete set of representatives of isomorphism classes of the indecomposable objects in $\mathrm{rep}^{-,i}(\widetilde{Q})$. For $r \geq 0$, set $\mathbb{Z}_r = \mathbb{Z}$ if $r = 0$, and $\mathbb{Z}_r = \{0, 1, \ldots r-1\}$ if $r > 0$.

3.11. Theorem. *Let A be a finite-dimensional connected elementary k-algebra with radical squared zero, and let $\pi : \widetilde{Q} \to Q$ be the minimal gradable covering and r the grading period of the ordinary quiver Q of A.*

(1) *The induced functor $\mathcal{F} : \mathrm{rep}^{-,i}(\widetilde{Q}) \to D^b(A)$ preserves isomorphism classes and indecomposability.*

(2) *The complexes $\mathcal{F}(M)^\bullet[s]$ with $M \in \mathrm{ind}^{-,i}(\widetilde{Q})$ and $s \in \mathbb{Z}_r$ are the non-isomorphic indecomposable objects in $RC^{-,b}(A\text{-proj})$.*

(3) *If $\mathrm{Hom}_{D^b(A)}(\mathcal{F}(M)^\bullet, \mathcal{F}(N)^\bullet[s]) \neq 0$ with $M, N \in \mathrm{ind}^{-,i}(\widetilde{Q})$ and $s \in \mathbb{Z}_r$, then $s = 0$ or 1.*

Proof. Statement (1) follows from Proposition 3.1(3) and Lemma 3.9. Let $M, N \in \mathrm{ind}^{-,i}(\widetilde{Q})$ and $s, t \in \mathbb{Z}_r$ such that $\mathcal{F}(M)^\bullet[s] \cong \mathcal{F}(N)^\bullet[t]$. Since G and E preserve isomorphism classes, we have $F(M)^\bullet[s] \cong F(N)^\bullet[t]$. By Lemma 3.10, $s = t$. Thus $F(M)^\bullet \cong F(N)^\bullet$. By Lemma 3.9, $M \cong N$, and hence $M = N$. This shows that the $\mathcal{F}(M)^\bullet[s]$ are pairwise non-isomorphic. Next let $X^\bullet$ be an indecomposable object in $D^b(A)$. In view of Proposition 3.1(3), $X^\bullet \cong E(G(Y^\bullet))$ for some indecomposable object $Y^\bullet$ in $RC^{-,b}(A\text{-proj})$. By Lemma 3.4, $Y^\bullet \cong F(L)^\bullet[s_0]$ with $L \in \mathrm{ind}^{-,i}(\widetilde{Q})$ and $s_0 \in \mathbb{Z}$. Now $s_0 \equiv s(\mathrm{mod} r)$

for some $s \in \mathbb{Z}_r$. Applying Lemma 3.10, we get $F(L)^\bullet[s_0] = F(L^\sigma)[s]$ for some $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$. Hence $X^\bullet \cong \mathcal{F}(L^\sigma)[s]$. This proves (2).

Assume that $\mathrm{Hom}_{D^b(A)}(\mathcal{F}(M)^\bullet, \mathcal{F}(N)^\bullet[s]) \neq 0$ with $M, N \in \mathrm{ind}^{-,i}(\widetilde{Q})$ and $s \in \mathbb{Z}_r$. If $r = 1$, then $\mathbb{Z}_r = \{0\}$. In particular, $s = 0$. Assume that $r \neq 1$. Then $0, 1 \in \mathbb{Z}_r$. Since G, E are full, $\mathrm{Hom}_{RC^{-,b}(A\text{-proj})}(F(M)^\bullet, F(N)^\bullet[s]) \neq 0$. By Lemma 3.5, $F(N)^\bullet[s] \cong F(N_1)^\bullet[t_1]$ with $N_1 \in \mathrm{ind}^{-,i}(\widetilde{Q})$ and $0 \le t_1 \le 1$. Noting that $t_1 \in \mathbb{Z}_r$, we get $s = t_1$ by (2). This completes the proof of the theorem.

Next we shall extend the functor $\mathcal{F}$ to an exact functor of triangulated categories $\widehat{\mathcal{F}} : D^b(\mathrm{rep}^{-,i}(\widetilde{Q})) \to D^b(A)$. Let $M^\bullet$ be a bounded complex in $\mathrm{rep}^{-,i}(\widetilde{Q})$. Applying F to each of the components of $M^\bullet$, we get a double complex $F(M^\bullet)^\bullet$ in A-proj as follows:

$$\begin{tikzcd}
 & \vdots & & \vdots & \\
\cdots \arrow[r] & F(M^{i+1})^j \arrow[u] \arrow[rr, "(-1)^{i+1} d^j_{F(M^{i+1})}"] & & F(M^{i+1})^{j+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & F(M^i)^j \arrow[u, "F(d^i_M)^j"] \arrow[rr, "(-1)^i d^j_{F(M^i)}"] & & F(M^i)^{j+1} \arrow[u, "F(d^i_M)^{j+1}"'] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}$$

which is clearly bounded. We then define $\widehat{F}(M^\bullet)^\bullet \in C^-(A\text{-pro})$ to be the total complex of the double complex $F(M^\bullet)^\bullet$. More explicitly, let s, t be integers such that $M^n \neq 0$ only if $s \le n \le t$. Then $\widehat{F}(M^\bullet)^n = \oplus_{i+j=n} F(M^i)^j$ and $d^n_{\widehat{F}(M^\bullet)}$ is given by a $(t-s+1) \times (t-s+1)$-matrix with (i,i)-entry being $(-1)^i d^{n-i}_{F(M^i)}$ for $s \le i \le t$, $(i, i+1)$-entry being $F(d^i_M)^{n-i}$ for $s \le i < t$, and all other entries being null. Using the Acyclic Assembly Lemma, see [23, (2.7.3)], we deduce easily that $\widehat{F}(M^\bullet)^\bullet$ has bounded cohomology and hence lies in $C^{-,b}(A\text{-proj})$. For a morphism $f^\bullet : M^\bullet \to N^\bullet$ in $C^b(\mathrm{rep}^{-,i}(\widetilde{Q}))$, setting $\widehat{F}(f^\bullet)^n = \oplus_{i+j=n} F(f^i)^j$, we get a morphism $\widehat{F}(f^\bullet)^\bullet = \{\widehat{F}(f)^n \mid n \in \mathbb{Z}\}$ of complexes from $\widehat{F}(M^\bullet)^\bullet$ to $\widehat{F}(N^\bullet)^\bullet$. This gives rise to a k-linear functor

$$\widehat{F} : C^b(\mathrm{rep}^{-,i}(\widetilde{Q})) \to C^{-,b}(A\text{-proj})$$

such that $\widehat{F}(M)^\bullet = F(M)^\bullet$ for object $M \in \mathrm{rep}^{-,i}(\widetilde{Q})$ and $\widehat{F}(f)^\bullet = F(f)^\bullet$ for morphism $f \in \mathrm{rep}^{-,i}(\widetilde{Q})$.

3.12. Lemma. *Let $M^\bullet$ be an object and $f^\bullet$ a morphism in $C^b(\mathrm{rep}^{-,i}(\widetilde{Q}))$.*

(1) $\widehat{F}(M^\bullet[1])^\bullet = \widehat{F}(M^\bullet)^\bullet[1]$, *and* $C^\bullet_{\widehat{F}(f^\bullet)} = \widehat{F}(C^\bullet_f)^\bullet$.

(2) $\widehat{F}(M^\bullet)^\bullet$ *is acyclic whenever* $M^\bullet$ *is acyclic.*

Proof. It is a routine verification that $\widehat{F}(M^\bullet[1])^\bullet = \widehat{F}(M^\bullet)^\bullet[1]$. Suppose now that $M^\bullet$ is acyclic. Since F is exact, the double complex $F(M^\bullet)^\bullet$ has exact columns, and hence its total complex $\widehat{F}(M^\bullet)^\bullet$ is acyclic; see, for example, [23, (2.7.3)]. Let now $f^\bullet : M^\bullet \to N^\bullet$ be a morphism of complexes. Assume that s, t are integers such that $M^n \neq 0$ or $N^n \neq 0$ only if $s \leq n \leq t$. Let n be a fixed integer. We see that $C^n_f = M^{n+1} \oplus N^n$, $F(C^n_f)^\bullet = F(M^{n+1})^\bullet \oplus F(N^n)^\bullet$, and $C^n_{\widehat{F}(f^\bullet)} = \widehat{F}(M^\bullet)^{n+1} \oplus \widehat{F}(N^\bullet)^n$. It is easy to check that

$$C^n_{\widehat{F}(f^\bullet)} = \oplus_{i=s}^t (F(M^i)^{n-i+1} \oplus F(N^i)^{n-i}) = \widehat{F}(C^\bullet_f)^n.$$

For $s \leq i, j \leq t$ and $U, V \in \{M, N\}$, let $g_{ij}(U, V)$ be the composite

$$F(U^i)^{n-i+1} \xrightarrow{q_i(U)} C^n_{\widehat{F}(f^\bullet)} \xrightarrow{d^n_{C_{\widehat{F}(f^\bullet)}}} C^{n+1}_{\widehat{F}(f^\bullet)} \xrightarrow{p_j(V)} F(V^j)^{n+1},$$

where $q_i(U)$ is the canonical injection and $p_j(V)$ is the canonical projection. A routine but tedious verification shows that

$$g_{ij}(U,V) = \begin{cases} (-1)^{i+1} d^{n-i+1}_{F(M^i)}, & \text{if } j = i, \quad (U,V) = (M,M); \\ -F(d^i_M)^{n-i+1}, & \text{if } j = i+1, \quad (U,V) = (M,M); \\ F(f^i)^{n-i+1}, & \text{if } j = i, \quad (U,V) = (M,N); \\ (-1)^i d^{n-i}_{F(N^i)}, & \text{if } j = i, \quad (U,V) = (N,N); \\ F(d^i_N)^{n-i}, & \text{if } j = i+1, \quad (U,V) = (N,N); \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, let $h_{ij}(U, V)$ be the composite

$$F(U^i)^{n+1-i} \xrightarrow{q'_i(U)} \widehat{F}(C^\bullet_f)^n \xrightarrow{d^n_{\widehat{F}(C^\bullet_f)}} \widehat{F}(C^\bullet_f)^{n+1} \xrightarrow{p'_j(V)} F(V^j)^{n+1},$$

where $q'_i(U)$ is the canonical injection, and $p'_j(V)$ is the canonical projection. Observing that $q_i(M)' : F(M^i)^{n-i+1} \to \widehat{F}(C^\bullet_f)^n$ factors through $F(C^{i-1}_f)^{n-i+1}$ and $p'_j(M) : \widehat{F}(C^\bullet_f)^{n+1} \to F(M^j)^{n-j+2}$ factors through $F(C^{j-1}_f)^{n-j+2}$, we get

$$h_{ij}(U,V) = \begin{cases} (-1)^{i-1} d^{n-i+1}_{F(M^i)}, & \text{if } j = i, \quad (U,V) = (M,M); \\ -F(d^i_M)^{n-i+1}, & \text{if } j = i+1, \quad (U,V) = (M,M); \\ F(f^i)^{n-i+1}, & \text{if } j = i, \quad (U,V) = (M,N); \\ (-1)^i d^{n-i}_{F(N^i)}, & \text{if } j = i, \quad (U,V) = (N,N); \\ F(d^i_N)^{n-i}, & \text{if } j = i+1, \quad (U,V) = (N,N); \\ 0, & \text{otherwise.} \end{cases}$$

This shows that $d^n_{C_{\widehat{F}(f^\bullet)}} = d^n_{\widehat{F}(C^\bullet_f)}$. Thus $C^\bullet_{\widehat{F}(f^\bullet)} = \widehat{F}(C^\bullet_f)^\bullet$. The proof of the lemma is completed.

In view of Lemma 3.12, $\widehat{F}$ sends exact triangles to exact ones. Further, if $f^\bullet : X^\bullet \to Y^\bullet$ is a quasi-isomorphism in $C^b(\mathrm{rep}^{-,i}(\widetilde{Q}))$, then $C_f^\bullet$ is acyclic. By Lemma 3.12, $C^\bullet_{\widehat{F}(f^\bullet)} = \widehat{F}(C_f^\bullet)^\bullet$ is acyclic. Hence $\widehat{F}(f^\bullet)^\bullet$ is a quasi-isomorphism in $C^{-,b}(A\text{-proj})$. As a consequence, there exist exact functors of triangulated categories $\widetilde{F}$ and $\widehat{\mathcal{F}}$ making the following diagram commutative:

$$\begin{tikzcd}
C^b(\mathrm{rep}^{-,i}(\widetilde{Q})) \arrow[r, "p_{\widetilde{Q}}"] \arrow[d, "\widehat{F}"'] & K^b(\mathrm{rep}^{-,i}(\widetilde{Q})) \arrow[r, "q_{\widetilde{Q}}"] \arrow[d, "\widetilde{F}"'] & D^b(\mathrm{rep}^{-,i}(\widetilde{Q})) \arrow[d, "\widehat{\mathcal{F}}"] \\
C^{-,b}(A\text{-proj}) \arrow[r, "p_A"] & K^{-,b}(A\text{-proj}) \arrow[r, "E"] & D^b(A),
\end{tikzcd}$$

where $p_{\widetilde{Q}}, p_A$ are projection functors and $q_{\widetilde{Q}}$ is the localizing functor. We call $\widehat{\mathcal{F}}$ the *functor induced* from the minimal gradable covering π. It is easy to see that $\widehat{\mathcal{F}}|_{\mathrm{rep}^{-,i}(\widetilde{Q})} = \mathcal{F}$.

REMARK. Note that $A = kQ/I$ with I being generated by the paths of length two is the Koszul dual of the Koszul algebra kQ. If Q is gradable, then the functor $\widehat{F}$ coincides with the classical Koszul duality; see [4].

3.13. THEOREM. *Let A be a finite-dimensional connected elementary k-algebra with radical squared zero. Let $\pi : \widetilde{Q} \to Q$ be the minimal gradable covering of the ordinary quiver Q of A, and let $\widehat{\mathcal{F}} : D^b(\mathrm{rep}^{-,i}(\widetilde{Q})) \to D^b(A)$ be the induced functor. Then $\widehat{\mathcal{F}}$ is a triangle-equivalence if and only if Q is gradable.*

Proof. Assume that Q has positive grading period r. Let M be an indecomposable object in $\mathrm{rep}^{-,i}(\widetilde{Q})$. By Lemma 3.10, there exists an indecomposable object N in $\mathrm{rep}^{-,i}(\widetilde{Q})$ such that $F(M)^\bullet[r] = F(N)^\bullet$. Thus $\widehat{\mathcal{F}}(M[r]) \cong \widehat{\mathcal{F}}(N)$. In particular, $\widehat{\mathcal{F}}$ is not an equivalence.

We note that the sufficiency follows from [4]. However, we present a short proof using our own approach. Assume that Q is gradable and take $\pi = \mathbf{1}_Q$. It follows from Lemma 3.4 that $\widehat{\mathcal{F}}$ is dense. Note that $\mathrm{rep}^{-,i}(Q) = \mathrm{rep}(Q)$ since Q is finite, and $C^{-,b}(A\text{-proj}) = C^b(A\text{-proj})$ since Q has no oriented cycle. By Theorem 3.12, it suffices to show that $\widehat{\mathcal{F}}$ induces a bijection from $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M, N[t])$ onto $\mathrm{Hom}_{D^b(A)}(\widehat{\mathcal{F}}(M), \widehat{\mathcal{F}}(N)[t])$, for $M, N \in \mathrm{rep}(Q)$ and $t \in \mathbb{Z}$. If $t \neq 0, 1$, then this follows from Theorem 3.11(3) and the fact that $\mathrm{rep}(Q)$ is hereditary. Assume first that $t = 0$. Consider the following commutative diagram:

$$\begin{CD}
\mathrm{Hom}_{C^b(\mathrm{rep}(Q))}(M,N) @>{\widehat{F}_{MN}}>> \mathrm{Hom}_{C^b(A\text{-proj})}(\widehat{F}(M)^\bullet, \widehat{F}(N)^\bullet)\\
@V{p_Q(M,N)}VV @VV{p_A(M,N)}V\\
\mathrm{Hom}_{K^b(\mathrm{rep}(Q))}(M,N) @>{\widetilde{F}_{MN}}>> \mathrm{Hom}_{K^b(A\text{-proj})}(\widetilde{F}(M)^\bullet, \widetilde{F}(N)^\bullet)\\
@V{q_Q(M,N)}VV @V{\cong}V{E_{MN}}V\\
\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M,N) @>{\widehat{\mathcal{F}}_{MN}}>> \mathrm{Hom}_{D^b(A\text{-proj})}(\widehat{\mathcal{F}}(M)^\bullet, \widehat{\mathcal{F}}(N)^\bullet).
\end{CD}$$

Since $\mathrm{rep}(Q)$ is a full subcategory of each of $C^b(\mathrm{rep}(Q))$, $K^b(\mathrm{rep}(Q))$, and $D^b(\mathrm{rep}(Q))$, both $p_Q(M,N)$ and $q_Q(M,N)$ are bijective. On the other hand, by Lemma 3.9, the functor $F : \mathrm{rep}(Q) \to RC^b(A\text{-proj})$ is fully faithful. Thus $\widehat{F}_{MN}$ is a bijection. Since Q is gradable, $\mathrm{Hom}_A(F(M)^{n+1}, F(N)^n) = 0$ for all integers n. Thus a morphism $F(M)^\bullet \to F(N)^\bullet$ is null-homotopic if and only if it is null. This implies that $p_A(M,N)$ is injective and hence bijective. As a consequence, $\widehat{\mathcal{F}}_{MN}$ is bijective. Consider now the case where $t = 1$. Let $\theta^\bullet : \widehat{\mathcal{F}}(M)^\bullet \to \widehat{\mathcal{F}}(N)^\bullet[1]$ be a morphism in $D^b(A\text{-mod})$, which embeds in an exact triangle $\widehat{\mathcal{F}}(N)^\bullet \xrightarrow{\phi^\bullet} Y^\bullet \xrightarrow{\psi^\bullet} \widehat{\mathcal{F}}(M)^\bullet \xrightarrow{\theta^\bullet} \widehat{\mathcal{F}}(N)^\bullet[1]$. By Theorem 3.11(3), the above triangle is isomorphic to an exact triangle

$$\widehat{\mathcal{F}}(N)^\bullet \xrightarrow{(\zeta^\bullet,\,\nu^\bullet)} \widehat{\mathcal{F}}(L_1)^\bullet \oplus Z^\bullet \xrightarrow{\binom{\xi^\bullet}{0}} \widehat{\mathcal{F}}(M)^\bullet \xrightarrow{\theta^\bullet} \widehat{\mathcal{F}}(N)^\bullet[1]\ ,$$

where $L_1 \in \mathrm{rep}(Q)$ and $Z^\bullet \in D^b(A\text{-mod})$. Since $(0, \mathbf{1}_{Z^\bullet})\binom{\xi^\bullet}{0} = 0$, there exists $\mu^\bullet : Z^\bullet \to F(N)^\bullet$ such that $\mu^\bullet(\zeta^\bullet, \nu^\bullet) = (0, \mathbf{1}_{Z^\bullet})$. In particular, $\mu^\bullet\nu^\bullet = \mathbf{1}_{Z^\bullet}$. As a consequence, $Z^\bullet$ is either null or a direct summand of $\widehat{\mathcal{F}}(N)^\bullet$. Since $\widehat{\mathcal{F}}$ preserves indecomposability, we see that $Z^\bullet \cong \widehat{\mathcal{F}}(L_2)$ for some $L_2 \in \mathrm{rep}(Q)$. Thus $Y^\bullet \cong \widehat{\mathcal{F}}(L)^\bullet$, where $L = L_1 \oplus L_2$. Now it follows from what we have proved that $\phi^\bullet = \widehat{\mathcal{F}}(f)^\bullet$ for some $f : N \to L$ in $\mathrm{rep}(Q)$. Let $N \xrightarrow{f} L \xrightarrow{g} U \xrightarrow{h} N[1]$ be an exact triangle in $D^b(\mathrm{rep}(Q))$. This induces a commutative diagram

$$\begin{CD}
\widehat{\mathcal{F}}(N)^\bullet @>{\phi^\bullet}>> Y^\bullet @>{\psi^\bullet}>> \widehat{\mathcal{F}}(M)^\bullet @>{\theta^\bullet}>> \widehat{\mathcal{F}}(N)^\bullet[1]\\
@| @V{\cong}VV @V{\cong}V{\eta^\bullet}V @|\\
\widehat{\mathcal{F}}(N)^\bullet @>{\widehat{\mathcal{F}}(f^\bullet)}>> \widehat{\mathcal{F}}(L)^\bullet @>{\widehat{\mathcal{F}}(g)^\bullet}>> \widehat{\mathcal{F}}(U)^\bullet @>{\widehat{\mathcal{F}}(h)^\bullet}>> \widehat{\mathcal{F}}(N)^\bullet[1]
\end{CD}$$

in $D^b(A\text{-mod})$. By what we have proved, $\eta^\bullet = \widehat{\mathcal{F}}(u)^\bullet$ with $u : M \to U$ an isomorphism in $\mathrm{rep}(Q)$. Therefore, $\theta^\bullet = \widehat{\mathcal{F}}(uh)^\bullet$. Moreover, if $\theta^\bullet = 0$, then $\widehat{\mathcal{F}}(h)^\bullet = 0$. Hence $F(g)^\bullet$ is a retraction. As a consequence, g is a retraction, that is $h = 0$. In particular, $uh = 0$. This proves that $\widehat{\mathcal{F}}$ induces a bijection from $\operatorname{Hom}_{D^b(\mathrm{rep}(Q))}(M, N[1])$ onto $\operatorname{Hom}_{D^b(A)}(\widehat{\mathcal{F}}(M)^\bullet, \widehat{\mathcal{F}}(N)^\bullet[1])$. The proof of the theorem is completed.

Let $X^\bullet$ be a non-zero bounded complex in an additive category. If s is the minimal integer such that $X^s \neq 0$ and t is the maximal integer such that $X^t \neq 0$, then the positive integer $t - s + 1$ is called the *width* of $X^\bullet$. It is well known that every indecomposable object in the bounded derived category of a finite-dimensional hereditary algebra is isomorphic to a complex of width one; see [12].

3.14. Corollary. *Let A be a finite-dimensional elementary k-algebra with radical non-zero but squared zero, and let Q be the ordinary quiver of A. If Q is gradable, then $D^b(A) \cong D^b(kQ^{\mathrm{op}})$ and every indecomposable object in $D^b(A)$ is isomorphic to a complex of width less than the number of grading classes of Q.*

Proof. Assume that Q is gradable. It follows from Theorem 3.13 that there exists a triangle-equivalence $\widehat{\mathcal{F}} : D^b(\mathrm{rep}(Q)) \to D^b(A)$. Now kQ^{op} is a finite dimensional hereditary k-algebra such that $D^b(kQ^{\mathrm{op}}\text{-mod})$ is triangle-equivalent to $D^b(\text{mod-}kQ)$, where mod-kQ is the category of finite dimensional right kQ-modules which is equivalent to $\mathrm{rep}(Q)$. Hence $D^b(\text{mod-}kQ)$ is triangle-equivalent to $D^b(\mathrm{rep}(Q))$. Moreover, since A is not semi-simple, the number of grading classes of Q is an integer $m > 1$. Let M be an indecomposable object in $\mathrm{rep}(Q)$. Then $\mathcal{F}(M)^\bullet$ is an indecomposable object in $D^b(A)$ which, by the definition, is of width $\leq m$. Assume that $\mathcal{F}(M)^\bullet$ is of the form:

$$\cdots \to 0 \to X^s \xrightarrow{d^s} X^{s+1} \xrightarrow{d^{s+1}} X^{s+2} \xrightarrow{d^{s+2}} \cdots \xrightarrow{d^{t-2}} X^{t-1} \xrightarrow{d^{t-1}} X^t \to 0 \to \cdots$$

with $X^s \neq 0$ and $X^t \neq 0$. If $s = t$, then $\mathcal{F}(M)^\bullet$ is of width 1 which is less than m. Otherwise, X^s is a direct sum of simple projective A-modules. Thus the indecomposability of $\mathcal{F}(M)^\bullet$ implies that d^s is a monomorphism. Let $p : X^{s+1} \to Y^{s+1}$ be the cokernel of d^s. Then $d^{s+1} = p\,\bar{d}^{s+1}$ for some A-linear map $\bar{d}^{s+1} : Y^{s+1} \to X^{s+1}$. It is now evident that $\mathcal{F}(M)^\bullet$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow Y^{s+1} \xrightarrow{\bar{d}^{s+1}} X^{s+2} \xrightarrow{d^{s+2}} \cdots \xrightarrow{d^{t-2}} X^{t-1} \xrightarrow{d^{t-1}} X^t \longrightarrow 0 \longrightarrow \cdots$$

which is of width $t - s < m$. The proof of the corollary is completed.

4. Auslander-Reiten theory in $D^b(A)$

The Auslander-Reiten theory applies in the bounded derived category of a finite dimensional algebra of finite global dimension; see, for example, [12]. It is well understood for hereditary algebras. The objective of this section is to show that this is also the case for algebras with radical squared zero.

We begin with a brief recall. Let $\mathfrak{A}$ be an additive k-category in which the morphism spaces are finite dimensional over k and every indecomposable object has an elementary local endomorphism algebra. One calls a left almost morphism $f : X \to Y$ in $\mathfrak{A}$ a *source morphism for* X provided that a factorization $f = fh$ holds only if h is an automorphism. In this case, X is indecomposable and f is unique up to isomorphism. In a dual manner, one defines the notion of a *sink morphism*. Suppose that every indecomposable object in $\mathfrak{A}$ admits a source morphism and a sink morphism. One defines the *Auslander-Reiten quiver* $\varGamma_{\mathfrak{A}}$ of $\mathfrak{A}$ as follows. The set of vertices is a complete set of representatives of isomorphism classes of the indecomposable objects in $\mathfrak{A}$. The number of arrows from a vertex X to a vertex Y is the multiplicity of Y as an indecomposable summand of the codomain of the source morphism for X, or equivalently, the multiplicity of X as an indecomposable summand of the domain of the sink morphism for Y. The connected components of the quiver $\varGamma_{\mathfrak{A}}$ are called the *Auslander-Reiten components* of $\mathfrak{A}$.

Assume that $\mathfrak{A}$ is abelian. A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathfrak{A}$ is called *almost split* if f is a source morphism, or equivalently, g is a sink morphism. In this case, one calls X the *Auslander-Reiten translate* of Z and write $X = \tau_{\mathfrak{A}} Z$. One says that $\mathfrak{A}$ *has almost split sequences* provided that every indecomposable object X in $\mathfrak{A}$ admits a source morphism which is a monomorphism whenever X is non-injective, and a sink morphism which is an epimorphism whenever X is non-projective. In this case $\tau_{\mathfrak{A}}$, called the *Auslander-Reiten translation* for $\mathfrak{A}$, is defined on all indecomposable non-projective objects and makes $\varGamma_{\mathfrak{A}}$ a translation quiver in the sense of [20, (2.1)].

Assume that $\mathfrak{A}$ is triangulated with a shift functor T. Recall that an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ with X, Z indecomposable is called an *almost split triangle ending with* Z provided that f is left almost split or g is right almost split, or equivalently, f is a source morphism and g is a sink morphism; see [11, (4.1)]. In this case, one calls X the *Auslander-Reiten translate* of Z and writes $X = \tau_{\mathfrak{A}} Z$. One says that $\mathfrak{A}$ has almost split triangles provided that every indecomposable object in $\mathfrak{A}$ is the ending term of an almost split triangle. In this case $\tau_{\mathfrak{A}}$, called the *Auslander-Reiten translation* for $\mathfrak{A}$, is defined on all indecomposable objects and makes $\varGamma_{\mathfrak{A}}$ a stable translation quiver.

For describing the shapes of Auslander-Reiten components, one needs a classical construction of a stable translation quiver $\mathbb{Z}\Delta$ from a quiver $\varDelta$ with no oriented cycle. The vertices of $\mathbb{Z}\Delta$ are (n, x) with $n \in \mathbb{Z}$ and $x \in \varDelta_0$. Each arrow $x \to y$ in $\varDelta$ induces, for each $n \in \mathbb{Z}$, two arrows $(n, x) \to (n, y)$ and $(n, y) \to (n+1, x)$ in $\mathbb{Z}\varDelta$. The set of arrows in $\mathbb{Z}\varDelta$ is formed by such induced arrows. The translation τ is defined by $\tau(n, x) = (n-1, x)$, $n \in \mathbb{Z}$. Denote by $\mathbb{N}\varDelta$ the full translation subquiver of $\mathbb{Z}\varDelta$ generated by (n, x) with $x \in \varDelta_0$ and $n \geq 0$, and by $\mathbb{N}^-\varDelta$ that generated by (n, x) with $x \in \varDelta_0$ and $n \leq 0$. If $\varDelta$ is a tree of type $\varOmega$, then $\mathbb{Z}\varDelta$ does not depend on the orientation of $\varDelta$ and we write

$\mathbb{Z}\Delta = \mathbb{Z}\Omega$. Finally, recall that a translation quiver is called a *stable tube* if it is isomorphic to $\mathbb{Z}\mathbb{A}_\infty/\!<\!\tau^s\!>$ for some $s \ge 1$.

For the rest of this section, let $A = kQ/I$, where Q is a finite connected quiver and I is the ideal in kQ generated by the paths of length two. Fix a minimal gradable covering $\pi : \widetilde{Q} \to Q$ of Q. Assume that A is of finite global dimension. Then Q contains no oriented cycle, and consequently, $\widetilde{Q}$ contains no infinite path. Thus the indecomposable projective and injective representations of $\widetilde{Q}$ are finite-dimensional, and consequently, $\mathrm{rep}^{-,i}(\widetilde{Q}) = \mathrm{rep}^b(\widetilde{Q})$. Now $\mathrm{rep}^b(\widetilde{Q})$ admits almost split sequences; see [6, (2.2)]. The shapes of the Auslander-Reiten components of $\mathrm{rep}^b(\widetilde{Q})$ are well described in case $\widetilde{Q}$ is finite or of type $\mathbb{A}_\infty^\infty$; see [7, 17, 18]. We shall generalize these results to a more general context. For each $a \in \widetilde{Q}_0$, let P_a and I_a be the associated indecomposable projective and injective representations. As usual, one sees easily that the inclusion map $q_a : \mathrm{rad}P_a \to P_a$ is the sink morphism for P_a, and the canonical projection $p_a : I_a \to I_a/\mathrm{soc}I_a$ is the source morphism for I_a. Thus, for $a, b \in \widetilde{Q}_0$, the number of arrows from P_a to P_b in $\Gamma_{\mathrm{rep}^b(\widetilde{Q})}$ is equal to the number of arrows from b to a in $\widetilde{Q}$. As a consequence, the full subquiver of $\Gamma_{\mathrm{rep}^b(\widetilde{Q})}$ generated by the P_a with $a \in \widetilde{Q}$ is isomorphic to $\widetilde{Q}^{\mathrm{op}}$, the opposite quiver of $\widetilde{Q}$. In particular, the P_a with $a \in \widetilde{Q}_0$ lie in the same connected component of $\Gamma_{\mathrm{rep}^b(\widetilde{Q})}$, called the *preprojective component*. Dually, the I_a with $a \in \widetilde{Q}_0$ lie in the same connected component of $\Gamma_{\mathrm{rep}^b(\widetilde{Q})}$, called the *preinjective component*, and generate a full subquiver isomorphic to $\widetilde{Q}^{\mathrm{op}}$. The Auslander-Reiten components of $\mathrm{rep}^b(\widetilde{Q})$ which are neither preprojective nor preinjective are called *regular*.

4.1. LEMMA. *Assume that $\widetilde{Q}$ is infinite but contains no infinite path. Then the Auslander-Reiten quiver of* $\mathrm{rep}^b(\widetilde{Q})$ *consists of the preprojective component which is of shape $\mathbb{N}\widetilde{Q}^{\mathrm{op}}$, the preinjective component which is of shape $\mathbb{N}^-\widetilde{Q}^{\mathrm{op}}$, and some regular components which are of shape $\mathbb{Z}\mathbb{A}_\infty$. Moreover, the number of regular components is* 2 *in case $\widetilde{Q}$ is of type $\mathbb{A}_\infty^\infty$.*

Proof. First of all, as argued in [17, (II.3, III.3)], the preprojective and preinjective components of $\Gamma_{\mathrm{rep}^b(\widetilde{Q})}$ are disjoint, and consequently, they are of shapes $\mathbb{N}\widetilde{Q}^{\mathrm{op}}$ and $\mathbb{N}^-\widetilde{Q}^{\mathrm{op}}$, respectively. Let $\mathcal{C}$ be a regular component of $\Gamma_{\mathrm{rep}^b(\widetilde{Q})}$. We claim that $\mathcal{C}$ contains no oriented cycle. Suppose on the contrary that there exists an oriented cycle Θ in $\mathcal{C}$. Let M be the direct sum of the modules appearing in Θ. Since $\widetilde{Q}$ contains no infinite path, there exists a finite connected convex subquiver $\widetilde{Q}_M$ of $\widetilde{Q}$ which supports the minimal projective and injective presentations of M in $\mathrm{rep}^b(\widetilde{Q})$. Note that the Auslander-Reiten translation for $\mathrm{rep}^b(\widetilde{Q})$ is given by DTr, the dual of the transpose; see [1]. In view of the construction of Dtr, one sees that Θ remains to be an oriented cycle in the Auslander-Reiten quiver of $\mathrm{rep}(\widetilde{Q}_M)$. It is then well known that $\widetilde{Q}_M$ is of Euclidean type; see, for example, [18]. Being connected and infinite, $\widetilde{Q}$ has a finite wild convex subquiver Σ containing $\widetilde{Q}_M$. Once again, Θ remains to be an oriented cycle in the Auslander-Reiten quiver of $\operatorname{rep}(\Sigma)$, and hence Σ is of Euclidean type. This contradiction confirms our claim. Being a stable translation quiver, $\mathcal{C}$ contains a section Δ, and hence $\mathcal{C} \cong \mathbb{Z}\Delta$; see [15, (2.1), (2.3),(2.4)]. Furthermore, Dtr preserves monomorphisms in $\operatorname{rep}^b(\widetilde{Q})$. Thus the dimension function yields a strict monotone additive function on $\mathcal{C}$, and consequently, Δ is either finite or of type $\mathbb{A}_\infty$; see [21]. Suppose that Δ is finite. Let N be the direct sum of the modules lying in Δ. Choose a finite connected convex subquiver Ω of $\widetilde{Q}$ which supports the minimal projective and injective presentations of N in $\operatorname{rep}^b(\widetilde{Q})$ and has more vertices than Δ does. It follows again from the construction of Dtr that Δ remains to be a section of a Auslander-Reiten component of $\operatorname{rep}(\Omega)$. In view of the shapes of the Auslander-Reiten components of $\operatorname{rep}(\Omega)$, we see that $\Omega \cong \Delta^{\rm op}$, which is absurd since Ω and $\Delta^{\rm op}$ do not have the same number of vertices. This proves the first part of the lemma, while the second part follows from [17, (III.3)]. The proof of the lemma is completed.

We now concentrate on the triangulated category $D^b(A)$. As did before, we choose arbitrarily a vertex a_0 in $\widetilde{Q}$ and put $\widetilde{Q}^n = \widetilde{Q}^n(a_0)$, for all $n \in \mathbb{Z}$. Let $F : \operatorname{rep}^{-,i}(\widetilde{Q}) \to RC^b(A\text{-proj})$ the functor induced from the covering $\pi : \widetilde{Q} \to Q$.

4.2. Lemma. *If $f : M \to N$ is a monomorphism in $\operatorname{rep}^{-,i}(\widetilde{Q})$, then every radical morphism $F(M)^\bullet \to X^\bullet$ in $RC^{-,b}(A\text{-proj})$ factors through $F(f)^\bullet$.*

Proof. Let $f : M \to N$ be a monomorphism in $\operatorname{rep}^{-,i}(\widetilde{Q})$. Consider a nonzero radical morphism $\phi^\bullet : F(M)^\bullet \to X^\bullet$ in $RC^{-,b}(A\text{-proj})$ with $X^\bullet$ indecomposable. By Lemma 3.5, we may assume that $X^\bullet = F(L)^\bullet[1]$ with $L \in \operatorname{rep}^{-,i}(\widetilde{Q})$. Fix an integer n, and write $\phi^n = (\phi^n(x,y))_{(x,y)\in \widetilde{Q}^n \times \widetilde{Q}^{n+1}}$, where $\phi^n(x,y)$ is a radical A-linear map from $P[x^\pi]\otimes M(x)$ to $P[y^\pi]\otimes L(y)$. Fix $(x,y) \in \widetilde{Q}^n \times \widetilde{Q}^{n+1}$. Since $Q_1(x^\pi, y^\pi) = \pi(\widetilde{Q}_1(x,y))$ by Lemma 1.5(3), we deduce from Lemma 3.2 that $\phi^n(x,y) = \sum_{\alpha \in \widetilde{Q}_1(x,y)} P[\alpha^\pi] \otimes g_\alpha$, where $g_\alpha \in \operatorname{Hom}_k(M(x), L(y))$. Since $f(x) : M(x) \to N(x)$ is injective, for each $\alpha \in \widetilde{Q}_1(x,y)$, there exists $h_\alpha \in \operatorname{Hom}_k(N(x), L(y))$ such that $g_\alpha = f(x)h_\alpha$. This yields a radical A-linear map $\psi^n(x,y) = \sum_{\alpha\in\widetilde{Q}_1(x,y)} P[\alpha^\pi]\otimes h_\alpha : P[x^\pi]\otimes N(x) \to P[y^\pi]\otimes L(y)$ such that $(\mathbf{1}_{P[x^\pi]} \otimes f(x))\psi^n(x,y) = \phi^n(x,y)$. Now $\psi^n = (\psi^n(x,y))_{(x,y)\in\widetilde{Q}^n\times\widetilde{Q}^{n+1}}$ is a radical A-linear map from $F(N)^n \to F(L)^{n+1}$ such that $\phi^n = F(f)^n\psi^n$. Since $\operatorname{rad}^2(A) = 0$, we see that $\psi^\bullet = \{\psi^n \mid n \in \mathbb{Z}\}$ is a morphism from $F(N)^\bullet$ to $F(L)^\bullet[1]$ such that $\phi^\bullet = F(f)^\bullet\psi^\bullet$. This completes the proof of the lemma.

4.3. Lemma. *The functor $F : \operatorname{rep}^{-,i}(\widetilde{Q}) \to RC^{-,b}(A\text{-proj})$ sends left almost split monomorphisms to left almost split monomorphisms, and right almost split epimorphisms to right almost split epimorphisms.*

Proof. Let $f : M \to N$ be a left almost split monomorphism in $\operatorname{rep}^{-,i}(\widetilde{Q})$. Since F is exact, $F(f)^\bullet$ is a monomorphism and is not a section by Lemma 3.6. Let $\phi^\bullet : F(M)^\bullet \to X^\bullet$ with $X^\bullet$ indecomposable be a nonzero morphism in $RC^{-,b}(A\text{-proj})$ which is not a section. If $\phi^\bullet$ is radical, then it factors through $F(f)^\bullet$ by Lemma 4.2. Otherwise, by Lemma 3.5, we may assume that $X^\bullet =$

$F(L)^\bullet$ with $L \in \mathrm{rep}^{-,i}(\widetilde{Q})$. By Lemma 3.6, $\phi^\bullet = F(g)^\bullet + \psi^\bullet$, where $g : M \to L$ is a morphism in $\mathrm{rep}^{-,i}(\widetilde{Q})$ which is not a section, and $\psi^\bullet$ is a radical morphism in $RC^{-,b}(A\text{-proj})$. Now g factors through f, and hence $F(g)^\bullet$ factors through $F(f)^\bullet$. Moreover, $\psi^\bullet$ factors through $F(f)^\bullet$ by Lemma 4.2. Thus, $\phi^\bullet$ factors through $F(f)^\bullet$. The proof of the lemma is completed.

Assume that A is of finite global dimension. We fix some notation for some special morphisms in $RC^b(A\text{-proj})$ associated to a vertex a in $\widetilde{Q}$. Assume that $a \in \widetilde{Q}^m$ for some integer m. First, write $p_a^\bullet = F(p_a)$ with $p_a : I_a \to I_a/\mathrm{soc}I_a$ the canonical projection and $q_a^\bullet = F(q_a)$ with $q_a : \mathrm{rad}P_a \to P_a$ the inclusion map. Since $I_a(x) = P_a(x)$ for all $x \in \widetilde{Q}^m$, we have $F(I_a)^m = F(P_a)^m = P[a^\pi]$. Moreover, $(\mathrm{rad}P_a)(y) = P_a(y)$ and $(I_a/\mathrm{soc}I_a)(y) = I_a(y)$ for all $y \neq a$. Consequently, $F(P_a)^n = F(\mathrm{rad}P_a)^n$ and $F(I_a)^n = F(I_a/\mathrm{soc}I_a)^n$, for all $n \neq m$. Now let $u_a^\bullet : F(I_a)^\bullet \to F(\mathrm{rad}P_a)^\bullet[1]$ be the morphism in $RC^b(A\text{-proj})$ with $u_a^m = d^m_{F(P_a)}$ and $u_a^n = 0$ for $n \neq m$, and $v_a^\bullet : F(I_a/\mathrm{soc}I_a)^\bullet \to F(P_a)^\bullet[1]$ the one such that $v_a^{m-1} = d^{m-1}_{F(I_a)}$ and $v_a^n = 0$ for $n \neq m-1$. Finally, define $w_a^\bullet : F(P_a)^\bullet \to F(I_a)^\bullet$ by $w_a^m = \mathbf{1}_{P[a^\pi]}$ and $w_a^n = 0$ for $n \neq m$.

4.4. Lemma. *Assume that A is of finite global dimension. Associated to each vertex a in $\widetilde{Q}$, there exists in $RC^b(A\text{-proj})$ a left almost split morphism $(p_a^\bullet, -u_a^\bullet) : F(I_a)^\bullet \to F(I_a/\mathrm{soc}I_a)^\bullet \oplus F(\mathrm{rad}P_a)^\bullet[1]$ and a right almost split one*

$$\begin{pmatrix} v_a^\bullet \\ q_a^\bullet[1] \end{pmatrix} : \ F(I_a/\mathrm{soc}I_a)^\bullet \oplus F(\mathrm{rad}P_a)^\bullet[1] \to F(P_a)^\bullet[1].$$

Proof. Let $a \in \widetilde{Q}^m$. We shall prove only the first part of the lemma since the second part follows dually. Let $\phi^\bullet : F(I_a)^\bullet \to X^\bullet$ with $X^\bullet$ indecomposable be a morphism in $RC^{-,b}(A\text{-proj})$ which is not a section. Assume first that $\phi^\bullet$ is radical. By Lemma 3.5, we may assume that $X^\bullet = F(N)^\bullet[1]$ with $N \in \mathrm{rep}^b(\widetilde{Q})$. Since F is exact, $RC^b(A\text{-proj})$ admits a short exact sequence

$$0 \longrightarrow F(S_a)^\bullet \xrightarrow{F(j_a)^\bullet} F(I_a)^\bullet \xrightarrow{p_a^\bullet} F(I_a/\mathrm{soc}I_a)^\bullet \longrightarrow 0,$$

where S_a is the simple representation concentrated on a and $j_a : S_a \to I_a$ is the inclusion map. For each n, define $\zeta^n : F(I_a)^n \to F(N)^{n+1}$ by $\zeta^n = 0$ for $n \geq m$ and $\zeta^n = \phi^n$ for $n < m$. Since the ζ^n are all radical, $\zeta^\bullet = \{\zeta^n \mid n \in \mathbb{Z}\}$ is a morphism from $F(I_a)^\bullet$ to $F(N)^\bullet[1]$ such that $F(j_a)^\bullet\zeta^\bullet = 0$. Thus $\zeta^\bullet$ factors through $p_a^\bullet$. Consider now $\theta^\bullet = \phi^\bullet - \zeta^\bullet$. Then $\theta^m = \phi^m$ and $\theta^n = 0$ for all $n \neq m$. Note that $u_a^m = (u_a^m(y))_{y \in \widetilde{Q}^{m+1}}$, where

$$u_a^m(y) = \sum\nolimits_{\alpha \in \widetilde{Q}_1(a,y)} P[\alpha^\pi] \otimes P_a(\alpha) : P[a^\pi] \otimes P_a(a) \to P[y^\pi] \otimes P_a(y).$$

Write $\theta^m = (\theta^m(y))_{y \in \widetilde{Q}^{m+1}}$ with $\theta^m(y) : P[a^\pi] \otimes P_a(a) \to P[y^\pi] \otimes N(y)$ a radical A-linear map. By Lemmas 1.5(3) and 3.2, $\theta^m(y) = \sum_{\alpha \in \widetilde{Q}_1(a,y)} P[\alpha^\pi] \otimes f_\alpha$ with $f_\alpha \in \mathrm{Hom}_k(P_a(a), N(y))$. Fix a vertex $y \in \widetilde{Q}^{m+1}$. Note that $\widetilde{Q}_1(a,y)$ is a k-basis of $(\mathrm{rad}P_a)(y)$. For each $\alpha \in \widetilde{Q}_1(a,y)$, denote by $g_\alpha : (\mathrm{rad}P_a)(y) \to N(y)$ the k-linear map which sends α to $f_\alpha(\varepsilon_a)$ and vanishes on the other arrows. Set $g_y = \sum_{\alpha\in\widetilde{Q}_1(a,y)} g_\alpha : (\mathrm{rad}P_a)(y) \to N(y)$. Since $P_a(\alpha)g_\alpha = f_\alpha$ and $P_a(\alpha)g_\beta = 0$ for $\beta \neq \alpha$, we have $\theta^m(y) = u_a^m(y)(\mathbf{1}_{P[y^\pi]} \otimes g_y)$. Now $\eta^m = (\mathbf{1}_{P[y^\pi]} \otimes g_y)_{y\in\widetilde{Q}^{m+1}}$ is an A-linear map from $F(\mathrm{rad}P_a)^{m+1}$ to $F(N)^{m+1}$ such that $\theta^m = u_a^m\eta^m$. Since $\mathrm{rad}P_a$ is projective with top $\oplus_{y\in\widetilde{Q}^{m+1}}(\mathrm{rad}P_a)(y)$, there exists an unique morphism $g : \mathrm{rad}P_a \to N$ in $\mathrm{rep}^b(\widetilde{Q})$ such that $g(y) = g_y$, for all $y \in \widetilde{Q}^{m+1}$. This yields a morphism $F(g)^\bullet : F(\mathrm{rad}P_a)^\bullet \to F(N)^\bullet$ in $RC^b(A\text{-proj})$ such that $F(g)^{m+1} = \eta^m$ and $F(g)^n = 0$ for $n \le m$. Thus $F(g)^\bullet[1]$ is a morphism in $RC^b(A\text{-proj})$ such that $\theta^\bullet = u_a^\bullet F(g)^\bullet[1]$. Thus, $\phi^\bullet = \theta^\bullet + \zeta^\bullet$ factors through the morphism $(u_a^\bullet, -p_a^\bullet)$.

Suppose now that $\phi^\bullet$ is not radical. By Lemma 3.5, we may assume that $X^\bullet = F(N)^\bullet$ with $N \in \mathrm{rep}^b(\widetilde{Q})$. By Lemma 3.6, $\phi^\bullet = F(f)^\bullet + \psi^\bullet$ with f a morphism in $\mathrm{rep}^b(\widetilde{Q})$ which is not a section, and $\psi^\bullet$ a radical morphism in $RC^b(A\text{-proj})$. Since f factors through $p_a : I_a \to I_a/\mathrm{soc}I_a$, we see that $F(f)^\bullet$ factors through $p_a^\bullet$. Being radical, $\psi^\bullet$ factors through $(p_a^\bullet, -u_a^\bullet)$ as shown previously. Consequently, $\phi^\bullet$ factors through $(u_a^\bullet, -p_a^\bullet)$. This completes the proof of the lemma.

We are now ready to describe the almost split triangles in $D^b(A)$ in case A is of finite global dimension.

4.5. Theorem. *Let A be a finite-dimensional elementary k-algebra with finite global dimension and radical squared zero. Let $\pi : \widetilde{Q} \to Q$ be a minimal gradable covering of the ordinary quiver Q of A, and let $\mathcal{F} : \mathrm{rep}^b(\widetilde{Q}) \to D^b(A)$ be the induced functor.*

(1) *Each almost split sequence $0 \to M \to N \to L \to 0$ in $\mathrm{rep}^b(\widetilde{Q})$ induces an almost split triangle $\mathcal{F}(M)^\bullet \to \mathcal{F}(N)^\bullet \to \mathcal{F}(L)^\bullet \to \mathcal{F}(M)^\bullet[1]$ in $D^b(A)$.*

(2) *For each vertex a in $\widetilde{Q}$, there exists in $D^b(A)$ an almost split triangle*

$$\mathcal{F}(I_a)^\bullet \to \mathcal{F}(I_a/\mathrm{soc}I_a)^\bullet \oplus \mathcal{F}(\mathrm{rad}P_a)^\bullet[1] \to \mathcal{F}(P_a)^\bullet[1] \to \mathcal{F}(I_a)^\bullet[1].$$

(3) *Every almost split triangle in $D^b(A)$ is isomorphic to a shift of some triangle stated in (1) or (2).*

Proof. Recall that $\mathcal{F}$ induces an exact functor $\widehat{\mathcal{F}} : D^b(\mathrm{rep}^b(\widetilde{Q})) \to D^b(A)$ with $\widehat{\mathcal{F}}|_{\mathrm{rep}^b(\widetilde{Q})} = \mathcal{F} = E \circ G \circ F$. Let $\eta : 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ be an almost split sequence in $\mathrm{rep}^b(\widetilde{Q})$. Then $M \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{\eta} M[1]$ is an exact triangle in $D^b(\mathrm{rep}^b(\widetilde{Q}))$. Thus $\widehat{\mathcal{F}}(M) \xrightarrow{\widehat{\mathcal{F}}(f)} \widehat{\mathcal{F}}(N) \xrightarrow{\widehat{\mathcal{F}}(g)} \widehat{\mathcal{F}}(L) \xrightarrow{\widehat{\mathcal{F}}(\eta)} \widehat{\mathcal{F}}(M)[1]$ is an exact triangle in $D^b(A)$. We deduce from Proposition 3.1(2) and Lemma 4.2 that $\widehat{\mathcal{F}}(f) = \mathcal{F}(f)$ is left almost split and $\widehat{\mathcal{F}}(g) = \mathcal{F}(g)$ is right almost split. This establishes (1). Now let a be a vertex in $\widetilde{Q}$, say $a \in \widetilde{Q}^m$ for some m. Let $C^\bullet$ be the mapping cone of $(p_a^\bullet, -u_a^\bullet) : F(I_a)^\bullet \to F(I_a/\mathrm{soc}I_a)^\bullet \oplus F(\mathrm{rad}P_a)^\bullet[1]$. Since

$$F(I_a/\mathrm{soc}I_a)^n \oplus F(\mathrm{rad}P_a)^n[1] = \begin{cases} F(\mathrm{rad}P_a)^{n+1} = F(P_a)^{n+1}, & \text{if } n \ge m; \\ F(I_a/\mathrm{soc}I_a)^n = F(I_a)^n, & \text{if } n < m, \end{cases}$$

we see that $C^n = F(P_a)^{n+1}$ for $n \ge m$, and $C^n = F(I_a)^{n+1} \oplus F(I_a)^n$ for $n < m$. Moreover, $d_C^n = -d^{n+1}_{F(P_a)}$ for $n \ge m$, $d_C^{m-1} = \binom{-d^m_{F(P_a)}}{0}$, and

$$d_C^n = \begin{pmatrix} -d^{n+1}_{F(I_a)} & \mathbf{1} \\ 0 & d^n_{F(I_a)} \end{pmatrix}$$

for $n < m-1$. Define a morphism $\xi^\bullet : C^\bullet \to F(P_a)^\bullet[1]$ in $C^b(A\text{-proj})$ by $\xi^n = 0$ for $n < m-1$, $\xi^{m-1} = (\mathbf{1}_{F(P_a)^m}, d^{m-1}_{F(I_a)})^T$, and $\xi^n = \mathbf{1}_{F(P_a)^{n+1}}$ for $n \ge m$; and a morphism $\eta^\bullet : F(P_a)^\bullet[1] \to C^\bullet$ such that $\eta^n = 0$ for $n < m-1$, $\eta^{m-1} = (\mathbf{1}_{F(P_a)^m}, 0)$, and $\eta^n = \mathbf{1}_{F(P_a)^{n+1}}$ for $n \ge m$. Then $\xi^\bullet\eta^\bullet = \mathbf{1}_{F(P_a)^\bullet[1]}$, while $\eta^\bullet\xi^\bullet$ is homotopic to $\mathbf{1}_{C^\bullet}$ via a contraction $g^\bullet$ defined by $g^n = 0$ for $n \ge m$, and

$$g^n = \begin{pmatrix} 0 & 0 \\ \mathbf{1}_{F(I_a)^n} & 0 \end{pmatrix},$$

for $n < m$. This proves that $\xi^\bullet$ is a homotopy equivalence. Consider now the diagram

$$\begin{tikzcd}
F(I_a)^\bullet \arrow[r, "{(p_a^\bullet, -u_a^\bullet)}"] \arrow[d, equal] & F(I_a/\mathrm{soc}I_a)^\bullet \oplus F(\mathrm{rad}P_a)^\bullet[1] \arrow[r, "j^\bullet"] \arrow[d, equal] & C^\bullet \arrow[r, "p^\bullet"] \arrow[d, "\xi^\bullet"] & F(I_a)^\bullet[1] \arrow[d, equal] \\
F(I_a)^\bullet \arrow[r, "{(p_a^\bullet, -u_a^\bullet)}"] & F(I_a/\mathrm{soc}I_a)^\bullet \oplus F(\mathrm{rad}P_a)^\bullet[1] \arrow[r, "\binom{v_a^\bullet}{q_a^\bullet[1]}"] & F(P_a)^\bullet[1] \arrow[r, "{w_a^\bullet[1]}"] & F(I_a)^\bullet[1],
\end{tikzcd}$$

in $RC^{-,b}(A\text{-proj})$, where $j^\bullet$ is the canonical injection, and $p^\bullet$ is the canonical projection. It is easy to verify that the square in the middle is commutative. Moreover, $p^\bullet$ is homotopic to $\xi^\bullet w_a^\bullet[1]$ via a contraction $h^\bullet$ defined by $h^n = 0$ for $n \ge m$, and $h^n = (0, \mathbf{1}_{F(I_a)^n})^T$ for $n < m$. Applying the projection functor $RC^{-,b}(A\text{-proj}) \to K^b(A)$ to the above diagram followed by the triangle-equivalence $E : K^b(A\text{-proj}) \to D^b(A)$, we get a commutative diagram in $D^b(A)$. In particular,

$$\mathcal{F}(I_a)^\bullet \longrightarrow \mathcal{F}(I_a/\mathrm{soc}I_a)^\bullet \oplus \mathcal{F}(\mathrm{rad}P_a)^\bullet[1] \longrightarrow \mathcal{F}(P_a)^\bullet[1] \longrightarrow \mathcal{F}(I_a)^\bullet[1],$$

is an exact triangle in $D^b(A)$ which, by Lemma 4.4 and Proposition 3.1(2), is an almost split triangle. This proves (2). Finally, (3) follows easily from the uniqueness of almost split triangles and the fact that $\mathrm{rep}^b(\widetilde{Q})$ has almost split sequences. The proof of the theorem is completed.

The previous result enables us to describe the shapes of the Auslander-Reiten components of $D^b(A)$ in case A is of finite global dimension.

4.6. THEOREM. *Let A be a finite-dimensional elementary k-algebra with finite global dimension and radical squared zero. Let $\pi : \widetilde{Q} \to Q$ be the minimal gradable covering of the ordinary quiver Q of A.*

(1) *If Q is gradable, then $\mathit{\Gamma}_{D^b(A)} \cong \mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$. As a consequence, an Auslander-Reiten component of $D^b(A)$ is either of shape $\mathbb{Z}Q^{\mathrm{op}}$ or $\mathbb{Z}\mathbb{A}_\infty$ or a stable tube.*

(2) *If Q is of positive grading period r, then $\mathit{\Gamma}_{D^b(A)}$ consists of r components of shape $\mathbb{Z}\widetilde{Q}^{\mathrm{op}}$ and some components of shape $\mathbb{Z}\mathbb{A}_\infty$ whose number is $2r$ in case Q is of type $\widetilde{\mathbb{A}}_n$.*

Proof. The first part of (1) is an immediate consequence of Theorem 3.13, while the second part is well-known; see [12, (5.6)]. Assume now that Q is of positive grading period r. By Lemma 4.1, $\mathit{\Gamma}_{\mathrm{rep}^b(\tilde{Q})}$ consists of the preprojective component $\mathcal{P}$, the preinjective component $\mathcal{I}$ and a set $\mathcal{R}$ of regular components. Let $\mathcal{F} : \mathrm{rep}^b(\widetilde{Q}) \to D^b(A)$ be the functor induced from π. For each integer i, denote by $\mathcal{I}[i]$ the set of complexes $\mathcal{F}(M)^\bullet[i]$ with $M \in \mathcal{I}$, by $\mathcal{P}[i]$ the set of complexes $\mathcal{F}(N)^\bullet[i]$ with $N \in \mathcal{P}$, and $\mathcal{R}[i]$ the set of complexes $\mathcal{F}(L)^\bullet[i]$ with L lying in some component in $\mathcal{R}$. By Theorem 3.11(2), the set of vertices of $\mathit{\Gamma}_{\mathrm{rep}^b(\tilde{Q})}$ is formed by the complexes lying in $\mathcal{I}[i]$, $\mathcal{R}[i]$ and $\mathcal{P}[i+1]$, $i \in \mathbb{Z}_r$.

Let $\mathcal{C}$ be a component in $\mathcal{R}$. For each $i \in \mathbb{Z}_r$, we deduce easily from Theorems 4.5(1) and 3.11(1) that the complexes $\mathcal{F}(M)^\bullet[i]$ with $M \in \mathcal{C}$ form an Auslander-Reiten component $\mathcal{C}_i$ of $D^b(A)$ which is isomorphic to $\mathcal{C}$ as a translation quiver. By Theorem 3.11(2), the $\mathcal{C}_i$ with $i \in \mathbb{Z}_r$ are r pairwise distinct components which are of shape $\mathbb{Z}\mathbb{A}_\infty$. If Q is type $\widetilde{\mathbb{A}}_n$, then $\mathcal{R}$ consists of two components, which induce $2r$ Auslander-Reiten components of shape $\mathbb{Z}\mathbb{A}_\infty$ of $\mathit{\Gamma}_{D^b(A)}$.

Next fix an integer $i \in \mathbb{Z}_r$. By Theorem 4.5(1) and 3.11(1), the complexes in $\mathcal{I}[i]$ generate a full subquiver of $\mathit{\Gamma}_{D^b(A)}$ of shape $\mathbb{N}\widetilde{Q}^{\mathrm{op}}$ which is closed under predecessors, while the complexes in $\mathcal{P}[i+1]$ generate a full subquiver of $\mathit{\Gamma}_{D^b(A)}$ of shape $\mathbb{N}^-\widetilde{Q}^{\mathrm{op}}$ which is closed under successors. Assume that $\mathcal{F}(M)^\bullet[i] = \mathcal{F}(N)^\bullet[i+1]$ for some $M \in \mathcal{I}$ and $N \in \mathcal{P}$. By Lemma 3.10, $N \cong M^\sigma$ for some $\sigma \in \mathrm{Aut}_\pi(\widetilde{Q})$. However, σ induces an auto-equivalence of $\mathrm{rep}^b(\widetilde{Q})$ and thus an automorphism of $\mathit{\Gamma}_{\mathrm{rep}^b(\tilde{Q})}$. In particular, $\mathcal{I}^\sigma$ is a connected component of $\mathit{\Gamma}_{\mathrm{rep}^b(\tilde{Q})}$ which is isomorphic to $\mathcal{I}$ as a translation quiver. In view of Lemma 4.1, we see that $\sigma(\mathcal{I}) = \mathcal{I}$. In particular, $N = M^\sigma \in \mathcal{I}$, a contradiction. Now we deduce from Theorem 4.7(2) that the complexes $\mathcal{F}(M)^\bullet[i]$ and $\mathcal{F}(N)^\bullet[i+1]$ form a component of $\mathit{\Gamma}_{D^b(A)}$ which is of shape $\mathbb{Z}\widetilde{Q}^{\mathrm{op}}$. This completes the proof of the theorem.

Recall that a finite dimensional k-algebra A is called *derived hereditary* or *piecewise hereditary* if $D^b(A)$ is triangle-equivalent to $D^b(H)$, where H is a finite-dimensional hereditary k-algebra.

4.7. Corollary. *Let A be a finite-dimensional elementary k-algebra with radical squared zero. Then A is derived hereditary if and only if the ordinary quiver of A is gradable.*

Proof. By Corollary 3.14, it suffices to show the necessity. Let Q be the ordinary quiver of A with minimal gradable covering $\widetilde{Q}$. Assume that $D^b(A) \cong D^b(H)$ with H a finite-dimensional k-algebra. Then A is of finite global dimension and the Auslander-Reiten quiver of $D^b(A)$ has components with finite sections. It follows then from Theorem 4.6 that $\widetilde{Q}$ is finite. By Proposition 1.8, Q is gradable. The proof of the corollary is completed.

5. The Derived type

Throughout this section, we assume that k is algebraically closed. Let A be a finite-dimensional k-algebra with bounded derived category $D^b(A)$. For $X^\bullet \in D^b(A)$, one calls the infinite vector

$$\underline{\mathrm{hdim}}(X^\bullet) = (\dim_k \mathrm{H}^n(X^\bullet))_{n\in\mathbb{Z}}$$

the *cohomology dimension vector* of $X^\bullet$, which has at most finitely many nonzero components. Denote by $\mathbb{N}^{(\mathbb{Z})}$ the set of vectors $\mathbf{h} = (h_n)_{n\in\mathbb{Z}}$ with $h_n \in \mathbb{N}$ such that $h_n = 0$ for all but finite many integers n.

One says that A is *derived finite* if $D^b(A)$ has only finitely many indecomposable objects up to shift and isomorphism. It is known that A is derived finite if and only if $D^b(A)$ is triangle-equivalent to $D^b(H)$ with H a hereditary k-algebra of Dynkin type; see [22].

Recall that A is *derived discrete* if, for any given $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$, there exist at most finitely many indecomposable objects up to isomorphism of cohomology dimension vector $\mathbf{h}$ in $D^b(A)$; see [22]. Moreover, we say that A is *strictly derived discrete* if it is derived discrete but not derived finite.

Next, consider the k-algebra $k[x]$ of polynomials in one variable. It is well known that the simple $k[x]$-modules up to isomorphism are $T_\lambda = k[x]/(x-\lambda)$, $\lambda \in k$. One says that A is *derived tame* if, for any given $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$, there exist bounded complexes $M_1^\bullet, \ldots, M_r^\bullet$ of A-$k[x]$-bimodules which are $k[x]$-free of finite rank such that all but finitely many (up to isomorphism) indecomposable objects of cohomology dimension vector $\mathbf{h}$ in $D^b(A)$ are of the form $M_i^\bullet \otimes_{k[x]} T_\lambda$ with $1 \le i \le r$ and $\lambda \in k$; compare [10]. We shall say that A is *strictly derived tame* if it is derived tame but not derived discrete.

Finally, let $\mathbb{F} = k < x, y >$ be the k-algebra of polynomials in two non-commuting variables, and denote by $\mathbb{F}$-mod the category of finite dimensional left $\mathbb{F}$-modules. One calls A *derived wild* if there exists a bounded complex $M^\bullet$ of A-$\mathbb{F}$-bimodules which are $\mathbb{F}$-free of finite rank such that the functor

$$M^\bullet \otimes_{\mathbb{F}} - : \mathbb{F}\text{-mod} \to D^b(A)$$

preserves indecomposability and isomorphism classes; compare [9]. It has been shown in [9] that A is either derived tame or derived wild, but not both.

5.1. LEMMA. *Let $(P^\bullet, d^\bullet)$ be a complex in $RC^-(A\text{-proj})$ of cohomology dimension vector $\mathbf{h} = (h_n)_{n\in\mathbb{Z}}$. Write $a = \dim_k A$. If $P^{r+1} = 0$ for some r, then*

$$\dim_k(P^n/\mathrm{rad}P^n) \le h_n + h_{n+1}a + \cdots + h_r a^{r-n}, \quad n \le r.$$

Proof. By hypothesis, $\mathrm{Im}d^n \subseteq \mathrm{rad}P^{n+1}$ for all n. Since $P^{r+1} = 0$, we have $\mathrm{H}^r(P^\bullet) = P^r/\mathrm{Im}d^{r-1}$. Hence $\dim_k(P^r/\mathrm{rad}P^r) \le \dim_k(P^r/\mathrm{Im}d^{r-1}) = h_r$. Assume that $n < r$ and that the statement holds for $n+1$. In particular, $\dim_k P^{n+1} \le a\dim_k(P^{n+1}/\mathrm{rad}P^{n+1}) \le h_{n+1}a + \cdots + h_r a^{r-n}$. Now

$$\begin{aligned} & \dim_k(P^n/\mathrm{rad}P^n) \le \dim_k(P^n/\mathrm{Im}\, d^{n-1}) \\ = \;& \dim_k(P^n/\mathrm{Ker}\, d^n) + \dim_k(\mathrm{Ker}\, d^n/\mathrm{Im}\, d^{n-1}) \\ = \;& \dim_k \mathrm{Im}\, d^n + \dim_k \mathrm{H}^n(P^\bullet) \\ \le \;& h_n + h_{n+1}a + \cdots + h_r a^{r-n}. \end{aligned}$$

This completes the proof of the lemma.

Let B be an arbitrary k-algebra. A B-representation M of a quiver Q consists of a family of right B-modules $M(a)$ with $a \in Q_0$ and a family of B-linear maps $M(\alpha) : M(a) \to M(b)$ with $\alpha : a \to b \in Q_1$. Clearly a B-representation of Q is also a k-representation. We say that M is *B-free of finite rank* if the $M(a)$ are all B-free such that $\oplus_{a\in Q_0} M(a)$ is of finite rank. In this case, for a finite dimensional left B-module U, denote by $M \otimes_B U$ the finite dimensional k-representation of Q such that $(M \otimes_B U)(a) = M(a) \otimes_B U$ for $a \in Q_0$, and $(M \otimes_B U)(\alpha) = M(\alpha) \otimes_B \mathbf{1}_U$ for $\alpha \in Q_1$.

5.2. THEOREM. *Let A be a finite dimensional k-algebra with radical squared zero, and let Q be the ordinary quiver A.*

(1) *If Q is of Dynkin type, then A is derived finite.*

(2) *If Q is non gradable of type $\widetilde{\mathbb{A}}_n$, then A is strictly derived discrete.*

(3) *If Q is gradable of Euclidean type, then A is strictly derived tame.*

(4) *If Q is wild, then A is derived wild.*

Proof. Let $\pi : \widetilde{Q} \to Q$ be the minimal gradable covering of Q. Let $F : \mathrm{rep}^{-,i}(\widetilde{Q}) \to RC^{-,b}(A\text{-proj})$ and $\mathcal{F} : \mathrm{rep}^{-,i}(\widetilde{Q}) \to D^b(A)$ be the induced functors as defined in Section 3. By Theorem 3.11, $\mathcal{F}$ preserves indecomposability and isomorphism classes, and every indecomposable object in $D^b(A)$ is isomorphic to a shift of some complex $\mathcal{F}(M)^\bullet$ with M an indecomposable object in $\mathrm{rep}^{-,i}(Q)$. If Q is of Dynkin type, then $\widetilde{Q} \cong Q$. Hence there exist only finitely many non-isomorphic indecomposable objects in $\mathrm{rep}(\widetilde{Q})$. Therefore, $D^b(A)$ has only finitely many indecomposable objects up to shift and isomorphism.

For proving the remaining cases, we need to introduce some notation. Fix a vector $\mathbf{h} = (h_n)_{n\in\mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$. Let r, s with $r \le s$ be integers such that $h_n = 0$ whenever $n > s$ or $n \le r$. Consider the full subquiver $\widetilde{Q}^{[r,s]}$ of $\widetilde{Q}$ generated by the vertices lying in the $\widetilde{Q}^n$ with $r \le n \le s$. Note that $\widetilde{Q}^{[r,s]}$ is finite and connected. Denote by $\mathcal{M}(\mathbf{h})$ the set of non-isomorphic indecomposable objects in $\mathrm{rep}^{-,i}(\widetilde{Q})$ whose images under $\mathcal{F}$ have cohomology dimension vector

$\mathbf{h}$. Let $M \in \mathcal{M}(\mathbf{h})$. Then $F(M)^\bullet$ is of cohomology dimension vector $\mathbf{h}$. Write $M^n = \oplus_{x\in Q^n} M(x)$ for all $n \in \mathbb{Z}$. If $n > s$, we deduce easily from Lemma 5.1 that $F(M)^n = 0$, and hence $M^n = 0$. Therefore,

$$\dim_k(M^{\geq r}) = \sum\nolimits_{r\leq n\leq s} \dim_k M^n = \sum\nolimits_{r\leq n\leq s} \dim_k F(M)^n/\mathrm{rad}F(M)^n.$$

Using again Lemma 5.1, we get a constant $c(\mathbf{h})$ independent of M such that $\dim_k(M^{\geq r}) \leq c(\mathbf{h})$. Moreover, we deduce from the proof of Proposition 3.8 that M is r-truncated injective. Thus $\mathcal{M}^{\geq r}(\mathbf{h}) = \{M^{\geq r} \mid M \in \mathcal{M}(\mathbf{h})\}$ is, by Proposition 2.4, a set of non-isomorphic indecomposable k-representations of $\widetilde{Q}^{[r,s]}$ of dimension $\leq c(\mathbf{h})$.

Consider now the case where Q is non-gradable of type $\widetilde{\mathbb{A}}_t$ with $t \geq 1$. By Proposition 1.8(1), $\widetilde{Q}$ is of type $\mathbb{A}_\infty^\infty$. As a consequence, $\widetilde{Q}^{[r,s]}$ is a quiver of type $\mathbb{A}_m$. Therefore, $\mathcal{M}^{\geq r}(\mathbf{h})$ is finite, and so is $\mathcal{M}(\mathbf{h})$ by Proposition 2.4(1). This shows that A is derived discrete. Furthermore, for each $n \geq 1$, there exists an indecomposable object V_n of dimension n in $\mathrm{rep}^b(\widetilde{Q})$, which gives rise to an indecomposable object of width n in $RC^{-,b}(A\text{-proj})$. Consequently, the $\mathcal{F}(V_n)$ with $n \geq 1$ are indecomposable objects in $D^b(A)$ such that $\mathcal{F}(V_n)$ is neither isomorphic to nor a shift of $\mathcal{F}(V_{n'})$ whenever $n \neq n'$. That is, A is not derived finite. This proves (2).

Next, we deal with the case where Q is gradable of Euclidean type. We may then assume that $\widetilde{Q} = Q$. It is well known that there exist $k[x]$-representations $M_1, \ldots, M_l$ of $\widetilde{Q}$ which are $k[x]$-free of finite rank such that, up to isomorphism, all but finitely many indecomposables objects in $\mathrm{rep}(\widetilde{Q})$ of dimension $\leq c(\mathbf{h})$ are of the form $M_i \otimes_{k[x]} T_\lambda$ with $\lambda \in k$ and $1 \leq i \leq l$. It is easy to see that $F(M_i \otimes_{k[x]} T_\lambda)^\bullet \cong F(M_i)^\bullet \otimes_{k[x]} T_\lambda$. Thus the $F(M_i)^\bullet$ with $1 \leq i \leq l$ are bounded complexes of A-$k[x]$-bimodules which are A-projective and $k[x]$-free of finite rank such that all but finitely many (up to isomorphism) indecomposable objects in $D^b(A)$ are of the form $F(M_i)^\bullet \otimes_{k[x]} T_\lambda$. Hence A is derived tame. Moreover, it is well known that there exists a vector $(n_x)_{x\in\widetilde{Q}_0}$ of positive integers and an infinite set $\mathcal{M}$ of non-isomorphic indecomposable objects M in $\mathrm{rep}(\widetilde{Q})$ such that $\dim_k M(x) = n_x$ for all $x \in \widetilde{Q}_0$. Therefore, there exist infinite many non-isomorphic complexes $F(M)^\bullet$ of the same cohomology dimension vector. This shows that A is not derived discrete.

Finally, suppose that Q is a wild quiver. By Proposition 1.8(2), $\widetilde{Q}$ has a finite connected full subquiver $\mathit{\Delta}$ of wild type. It is then well known that there exists an $\mathbb{F}$-representation N of $\mathit{\Delta}$ which is $\mathbb{F}$-free of finite rank such that

$$N \otimes_{\mathbb{F}} - : \mathbb{F}\text{-mod} \to \mathrm{rep}(\mathit{\Delta})$$

preserves indecomposability and isomorphism classes. Then $F(N)^\bullet$ is a bounded complex of A-$\mathbb{F}$-bimodules which are A-projective and $\mathbb{F}$-free of finite rank such that

$$F(N)^\bullet \otimes_{\mathbb{F}} - : \ \mathbb{F}\text{-mod} \to D^b(A)$$

preserves indecomposability and isomorphism classes. This completes the proof of the theorem.

Acknowledgement. The first author thanks the support of Fondo Sectorial SEP-CONACYT via the project 43374F, while the second author is supported in part by the Natural Sciences and Engineering Research Council of Canada and is a (985-project) guest researcher at Beijing Normal University, China.

References

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebra*, Cambridge studies in advanced mathematics **36** (Cambridge University Press, Cambridge, 1995).

[2] R. Bautista, *Irreducible morphisms and the radical of a category*, An. Inst. Mat. Univ. Autónoma México **22** (1982) 83-135.

[3] R. Bautista, M. J. Souto Salorio and R. Zuazua, *Almost split sequences for complexes of fixed size*, J. Algebra **287** (2005) 140-168.

[4] A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996) 473-527.

[5] V. I. Bekkert and H. A. Merklen, *Indecomposables in derived categories of gentle algebras*, Algebr. Represent. Theory **6** (2003) 285-302.

[6] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. **65** (1982) 331-378.

[7] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Memoirs Amer. Math. Soc. **173** (1976).

[8] P. W. Donovan and M. R. Freislich, *The representation theory of finite graphs and associate algebras*, Carleton Mathematics Lecture Notes **5** (Carleton University, Ottawa, 1973).

[9] Yu. A. Drozd, *Derived tame and derived wild algebras*, Algebra Discrete Math. **1** (2004) 57-74.

[10] Ch. Geiss and H. Krause, *On the notion of derived tameness*, J. Algebra Appl. **1** (2002) 133-157.

[11] D. Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987) 339-389.

[12] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Mathematical Society Lecture Note Series **119** (Cambridge University Press, Cambridge, 1988).

[13] D. Happel, *Auslander-Reiten triangles in derived categories of finite dimensional algebras*, Proc. Amer. Math. Soc. **112** (1991) 641-648.

[14] H. Krause, *A duality between complexes of right and left modules*, Representations of Algebras, Vol. I. Proceedings of the Ninth International Conference, Beijing, Beijing Normal University Press (2000), 87-97.

[15] S. Liu, *Shapes of connected components of the Auslander-Reiten quiver of artin algebras*, Representation theory of algebras and related topics (Mexico City, 1994), Canad. Math. Soc. Conf. Proc. **19** (Amer. Math. Soc., Providence, RI, 1996) 109-137,

[16] L. A. Nazarova, *Representations of quivers of infinite type*, Math. USSR. Izv. Ser. Math. **37** (1973) 752-791.

[17] I. Reiten and M. Van den Bergh, *Notherian hereditary categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002) 295 - 366.

[18] C. M. Ringel, *Finite dimensional hereditary algebras of wild representation type type*, Math. Z. **161** (1978) 236 - 255.

[19] C. M. Ringel, *Infinite dimensional Representations of Finite Dimensional Hereditary Algebras.* Instituto Nazionale di Alta Matematica Francesco Severi. Symposia Mathematica Vol XXIII (1979) 321-412.

[20] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics **1099** (Springer, Berlin, 1984).

[21] C. M. Ringel, *The regular components of the Auslander-Reiten quiver of a tilted algebra*, Chinese Ann. Math. Ser. B **9** (1988) 1 - 18.

[22] D. Vossieck, *The algebras with discrete derived category*, J. Algebra **243** (2001) 168-176.

[23] C. A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics **38** (Cambridge University Press, Cambridge, 1994)

Bautista Raymundo, Instituto de Matematicas, UNAM, Unidad Morelia, Apartado Postal 61-3, 58089 Morelia, Mexico.
Email: raymundo@matmor.unam.mx

Shiping Liu, Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1 and School of Mathematics, Beijing Normal University, Beijing, China.
Email: shiping.liu@usherbrooke.ca