Loops, L-spaces, and left-orderability

Joint work with Jonathan Hanselman

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Prelude: The motivating theorem

Theorem

Suppose Y is a closed, orientable three-manifold admitting a decomposition $Y \cong M_1 \cup M_2$ where the M_i are Seifert fibred spaces with torus boundary.

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(1) *Y is an L-space*

(2) *$\pi_1(Y)$ cannot be left-ordered*

(3) *Y does not admit a co-orientable taut foliation*

Prelude: The motivating theorem

Theorem [Boyer-Clay, Hanselman-W.]

Suppose Y is a closed, orientable three-manifold admitting a decomposition $Y \cong M_1 \cup M_2$ where the M_i are Seifert fibred spaces with torus boundary. Then the following are equivalent:

(1) *Y is an L-space*
(that is, Y it has simplest-possible Heegaard Floer homology)

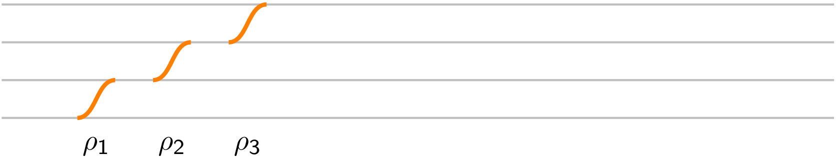
(2) *$\pi_1(Y)$ cannot be left-ordered*
(that is, the fundamental group does not act nicely on $\mathbb{R}$)

(3) *Y does not admit a co-orientable taut foliation*
(that is, no foliation by surfaces admitting a closed loop meeting every leaf transversally)

LOOPS

A simple algebra...

Consider an algebra $\mathcal{A}$ generated (over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$) by ρ_1, ρ_2, ρ_3:



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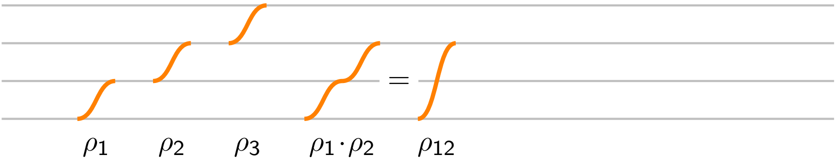

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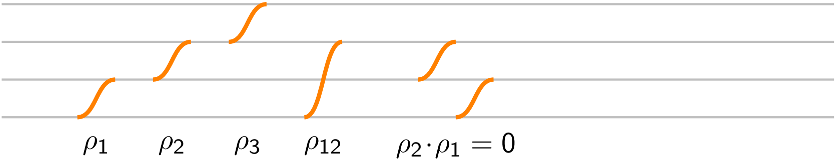

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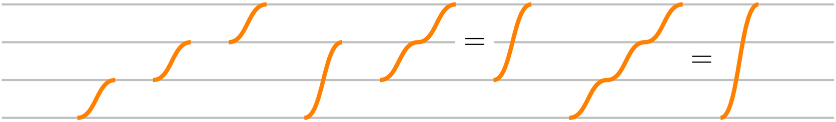

ρ_1 ρ_2 ρ_3 ρ_{12} $\rho_2 \cdot \rho_3$ ρ_{23} $\rho_1 \cdot \rho_2 \cdot \rho_3$ ρ_{123}

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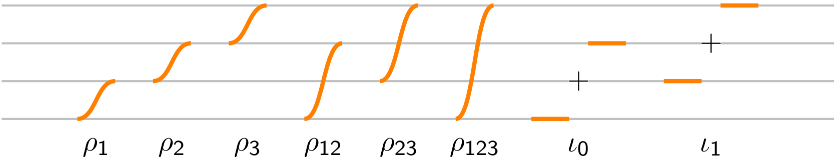
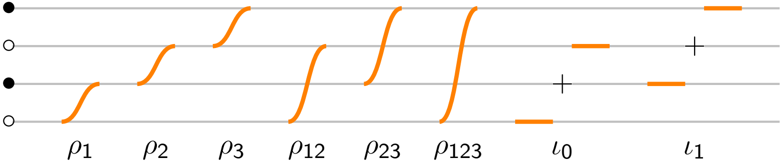
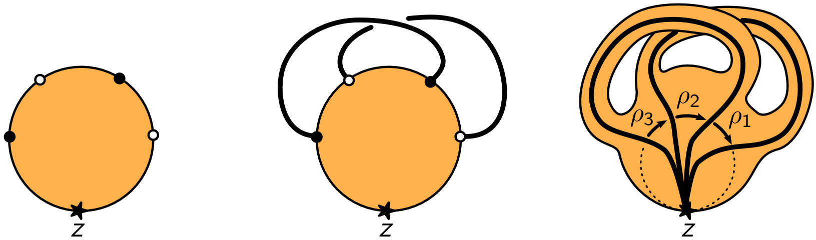


...associated with a torus.





$\mathcal{A}$-decorated graphs

An $\mathcal{A}$-decorated graph is a directed graph with

- Vetex set labeled by one of the idempotents ι_0 or ι_1; and
- Edge set labeled by one of the algebra elements ρ_1, ρ_2, ρ_3, ρ_{12}, ρ_{23}, or ρ_{123} (consistent with the edge-orientations).

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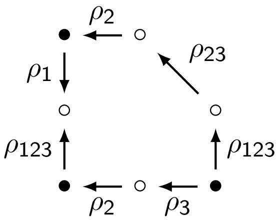
A loop is a **valence 2** $\mathcal{A}$-decorated graph – subject to certain restrictions.

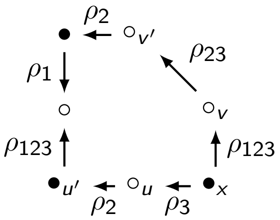


Differential modules

Any loop describes a (left) differential module over $\mathcal{A}$: Consider the $\mathbb{F}$-vector space generated by the vertex set; the differential determined by the edge set.

For example:

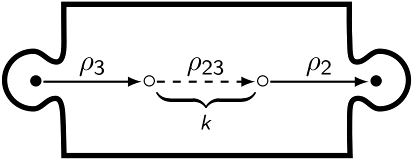
$$
\begin{aligned}
\partial(x) &= \rho_3 \cdot u + \rho_{123} \cdot v \\
\partial^2(x) &= \rho_3 \cdot \partial(u) + \rho_{123} \cdot \partial(v) \\
&= \rho_3\rho_2 \cdot u' + \rho_{123}\rho_{23} \cdot v' \\
&= 0
\end{aligned}
$$



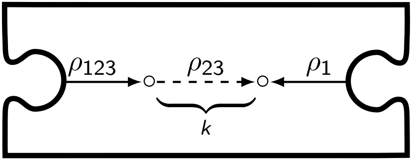
Consequently: $\partial^2 = 0$ places an *a priori* restriction on loops.

Puzzle-piece restrictions

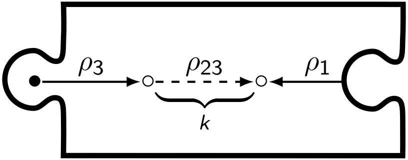
$a_k =$



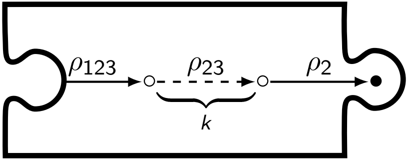
$b_k =$

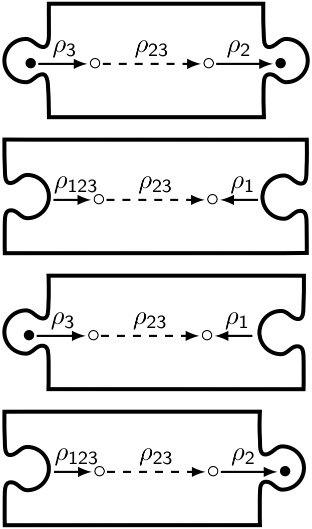


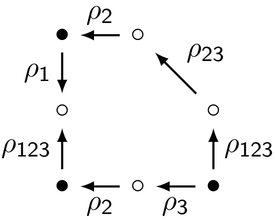
$c_k =$



$d_k =$







This loop, as a cyclic word:

$$(a_1 b_1 \bar{d}_2) = (d_2 \bar{b}_1 \bar{a}_1)$$

Heegaard Floer theory

Given a closed, orientable three-manifold Y, Heegaard Floer homology assigns a chain complex $\widehat{\mathrm{CF}}(Y)$ (over $\mathbb{F}$) that depends on a choice of Heegaard splitting of Y. The homology $\widehat{\mathrm{HF}}(Y)$ is an invariant of Y.

If $Y \cong M_1 \cup M_2$ (where ∂M_i is a torus) then bordered Heegaard Floer homology provides:

$$\widehat{\mathrm{CF}}(Y) \cong \widehat{\mathrm{CFA}}(M_1) \boxtimes \widehat{\mathrm{CFD}}(M_2)$$

Heegaard Floer theory

More precisely, suppose $Y \cong M_1 \cup_h M_2$ where the homeomorphism h is determined by $\alpha_1 \mapsto \beta_2$ and $\beta_1 \mapsto \alpha_2$. Write this instead as

$$Y \cong (M_1, \alpha_1, \beta_1) \cup (M_2, \alpha_2, \beta_2).$$

Each triple (M_i, α_i, β_i) is a *bordered* three-manifold, meaning $\langle \alpha_i, \beta_i \rangle \cong H_1(\partial M_i; \mathbb{Z})$. Now

$$\widehat{\mathrm{CF}}(Y) \cong \widehat{\mathrm{CFA}}(M_1, \alpha_1, \beta_1) \boxtimes \widehat{\mathrm{CFD}}(M_2, \alpha_2, \beta_2)$$

where

$\widehat{\mathrm{CFA}}(M_1, \alpha_1, \beta_1)$ is a right A_∞-module over $\mathcal{A}$; and

$\widehat{\mathrm{CFD}}(M_2, \alpha_2, \beta_2)$ is a left differential module over $\mathcal{A}$.

Loop-type manifolds

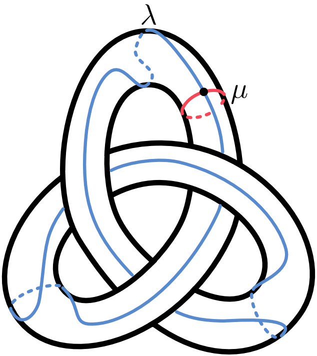
Definition

A bordered three-manifold (M, α, β) is loop-type if $\widehat{\mathrm{CFD}}(M, \alpha, \beta)$ is (up to homotopy) described by a loop.

Remark

It's not obvious, but this is not dependent on the bordered structure: (M, α, β) is loop-type if and only if (M, α', β') is.

Example: The left-hand trefoil



Let K be the left-hand trefoil and set

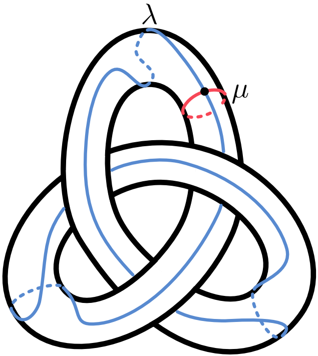
$$M = S^3 \setminus \nu(K)$$

Consider the bordered manifold

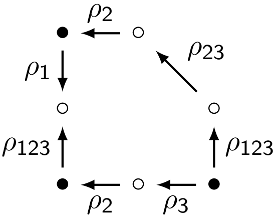
$$(M, \mu, \lambda)$$

where μ is the knot meridian and λ is the Seifert longitude.

Example: The left-hand trefoil



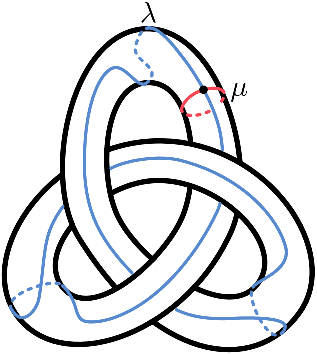
Then $\widehat{\mathrm{CFD}}(M, \mu, \lambda)$ is described by the loop



which is equivalent to the cyclic word

$$(a_1 b_1 \bar{d}_2) = (a_1 b_1 c_{-2})$$

Example: The left-hand trefoil



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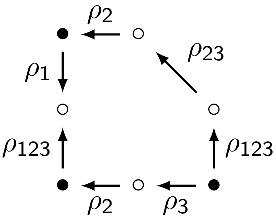
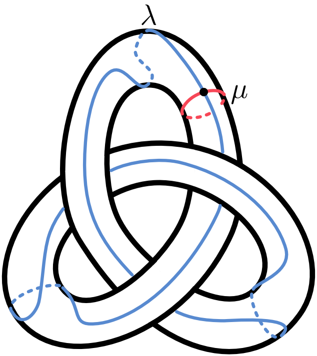


Note: Typically sensitive to the bordered structure:

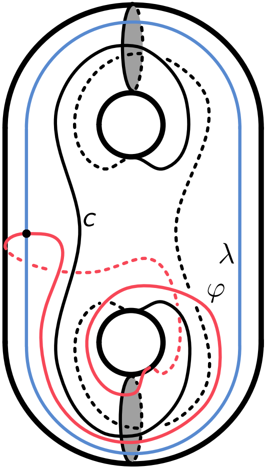
$$\widehat{\mathrm{CFD}}(M, \mu, \lambda - n\mu) = (a_1 b_1 c_{n-2})$$

Example: The left-hand trefoil



Remark: K is a torus knot, that is, K is isotopic to a regular fibre in a Seifert fibration of S^3. As a result, M admits a Seifert structure with base orbifold $D^2(2,3)$. This structure restricts to a foliation of ∂M by circles isotopic to the slope $\lambda - 6\mu$.

Example: The twisted I-bundle over the Klein bottle



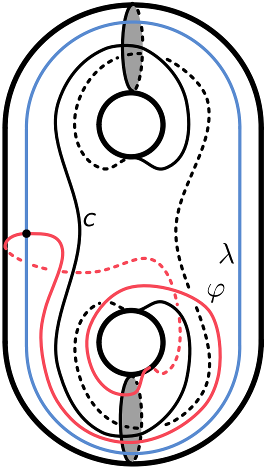
The twisted I-bundle over the Klein bottle N may be constructed as follows:

Attach a $D^2 \times I$ to the genus two handlebody by a homeomorphism sending $\partial D^2 \times \{\mathrm{pt}\}$ to the curve c.

Check: The fundamental group $\pi_1(N)$ has presentation

$$\langle a, b | a^2 b^2 \rangle$$

Example: The twisted I-bundle over the Klein bottle



$$\pi_1(N) \cong \langle a, b | a^2 b^2 \rangle$$

Exersises:

(1) ∂N is a torus;

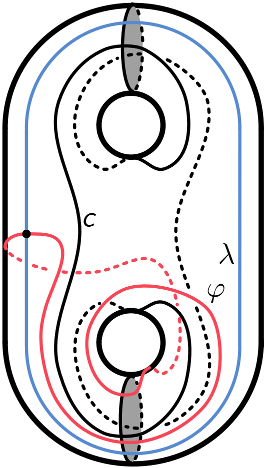
(2) N is Seifert fibred over $D^2(2,2)$ with regular fiber $\varphi \simeq [b^2]$; and

(3) N is Seifert fibred over a Mobius strip with regular fiber $\lambda \simeq [ab]$.

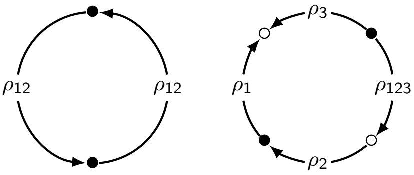
Consider the bordered manifold

$$(N, \varphi, \lambda)$$

Example: The twisted I-bundle over the Klein bottle

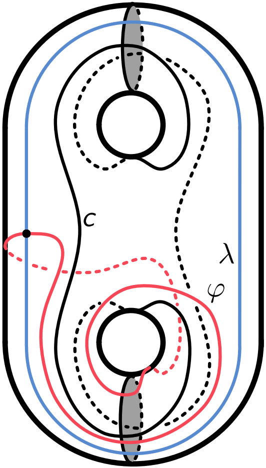


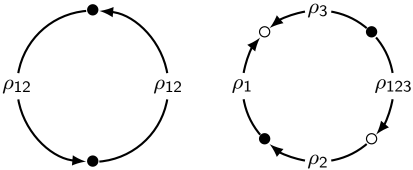
Then $\widehat{\mathrm{CFD}}(N, \varphi, \lambda)$ is represented by



and the components of this loop may be represented by $(d_0 d_0) = (ee)$ and $(d_1 d_{-1}) = (d_1 \bar{c}_1)$.

Example: The twisted I-bundle over the Klein bottle





Interesting feature:

Proposition

$\widehat{\mathrm{CFD}}(N, \varphi, \lambda) \cong \widehat{\mathrm{CFD}}(N, \varphi + n\lambda, \lambda)$

Key first step

Theorem [Hanselman-W.]

Any Seifert fibred rational homology solid torus is loop-type. In particular, if M is Seifert fibred over a disk or a Mobius strip, it is loop-type.

L-SPACES

Manifolds with simple Heegaard Floer homology

For rational homology spheres: $\dim \widehat{\mathrm{HF}}(Y) \geq |H_1(Y;\mathbb{Z})|$

Definition

An L-space is rational homology sphere for which

$$\dim \widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|$$

Examples:

(1) Lens spaces;

(2) manifolds with finite fundamental group; and

(3) the two fold branched cover of any alternating knot.

An interesting property

Examples:

(1) Lens spaces;

(2) manifolds with finite fundamental group; and

(3) the two fold branched cover of any alternating knot.

Theorem [Ozsváth-Szabó]

L-spaces do not admit co-orientable taut foliations.

L-space slopes

Let M be a three-manifold with torus boundary. A slope is an isotopy class of an essential, simple closed curve in ∂M. Let $M(\gamma)$ denote the closed manifold resulting from Dehn filling along γ. Define

$$\mathcal{L}_M = \{\gamma \mid M(\gamma) \text{ is an L-space}\}$$

Problem: Given M, describe the set $\mathcal{L}_M$.

Note that $\mathcal{L}_M$ may be identified with *some* subset of $\mathbb{Q} \cup \{\frac{1}{0}\}$. In particular, given (M, α, β), orient each curve and set

$$p\alpha + q\beta \quad \longleftrightarrow \quad \frac{p}{q}$$

$\mathcal{L}_M$ when M is loop-type

The idea is to develop a *loop calculus*, in order to prove things like:

Proposition

Suppose $\widehat{\mathrm{CFD}}(M, \alpha, \beta)$ *is a loop. Then* $M(\alpha)$ *is an L-space if and only if each connected component of the associated graph can be expressed as a cyclic word with at least one* d_k *and no* c_k *(where* k *ranges over the integers).*

Check:

(1) The left-hand trefoil exterior (M, μ, λ) *had* $(d_2 b_1 a_1)$*, and indeed* $M(\mu) \cong S^3$*.*

(2) The twisted I*-bundle over the Klein bottle* (N, φ, λ) *had* $(d_0 d_0)(d_1 d_{-1})$*, and indeed* $N(\varphi) \cong RP^3 \# RP^3$*.*

The effect of a Dehn twist

Proposition

Given a cyclic word in $\{a_k, b_k, c_k, d_k\}$ *representing the differential module* $\widehat{\mathrm{CFD}}(M, \alpha, \beta)$*, the effect of a Dehn twist producing* $\widehat{\mathrm{CFD}}(M, \alpha, \alpha + \beta)$ *is obtained by*

$$a_k \mapsto a_k, \quad b_k \mapsto b_k, \quad c_k \mapsto c_{k-1}, \quad d_k \mapsto d_{k+1}$$

These results form the basis for a complete characterization of $\mathcal{L}_M$ when M is loop-type.

A gluing result

Theorem [Hanselman-W.]

Suppose M_1 and M_2 are loop-type manifolds (other than the solid torus), and consider three-manifold $Y \cong M_1 \cup_h M_2$. If for each slope γ on ∂M either $\gamma \in \mathcal{L}^\circ_{M_1}$ or $h(\gamma) \in \mathcal{L}^\circ_{M_2}$ then Y is an L-space.

Sample calculation:

Theorem [Boyer-Gordon-W.]

Let N be the twisted I-bundle over the Klein bottle and suppose $Y \cong N \cup N$ is a rational homology sphere. Then Y is an L-space.

LEFT-ORDERABILITY

What is an L-space?

Let Y be an closed, orientable, irreducible three-manifold.

Conjecture [Boyer-Gordon-W.]

Y is an L-space if and only if $\pi_1(Y)$ is not left-orderable.

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Definition

A group G is left-orderable if there is a subset $\mathcal{P} \subset G$ satisfying:

- $G = \mathcal{P} \amalg \{1\} \amalg \mathcal{P}^{-1}$;
- $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}$; *and*
- $\mathcal{P} \neq \emptyset$

The set $\mathcal{P}$ is called a positive cone.

Example: $Homeo^+(\mathbb{R})$ is left-orderable

Let $Homeo^+(\mathbb{R})$ be the group of order/orientation preserving homeomorphisms of $\mathbb{R}$.
Choose a countable dense subset $X = \{x_1, x_2, x_3, \ldots\} \subset \mathbb{R}$. Set

$$f \in \mathcal{P}_X \iff f(x_n) > x_n \textit{ and } f(x_i) = x_i \textit{ for } i < n$$

Then $\mathcal{P}_X$ is a positive cone.

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Then $\mathcal{P}_X$ is a positive cone.

Theorem [Hölder]

Let G be a countable group. Then G is left-orderable if and only if $G \hookrightarrow Homeo^+(\mathbb{R})$.

Three-manifold groups

Theorem [Howie-Short, Boyer-Rolfsen-Wiest]

If Y is a compact, orientable, irreducible three-manifold then $\pi_1(Y)$ is left-orderable if and only if $\pi_1(Y)$ surjects to a left-orderable group.

Theorem [Boyer-Rolfsen-Wiest]

If Y is a closed Seifert fibred space, the following are equivalent:

(1) *Y admits a co-orientable taut foliation;*

(2) *Y admits an $\mathbb{R}$-covered foliation;*

(3) *$\pi_1(Y)$ is left-orderable*

Final steps

As a consequence of the gluing theorem:

Theorem [Hanselman-W.]

Let M be a loop type manifold, and let M' be a loop-type manifold for which all non-longitudinal fillings are L-space. For any slope γ the following are equivalent:

(1) $\gamma \in \mathcal{L}^{\circ}_{M}$

(2) *$M \cup_h M'$ is an L-space, where $h(\gamma) = \lambda'$*

(3) *$M \cup_h N$ is an L-space, where N is the twisted I-bundle over the Klein bottle and $h(\gamma) = \lambda$*

Remark: The N_t appearing in Boyer-Clay all satisfy the condition that non-longitudinal fillings are L-space.

Reprise

Theorem

Suppose Y is a closed, orientable three-manifold admitting a decomposition $Y \cong M_1 \cup M_2$ where the M_i are Seifert fibred spaces with torus boundary. Then the following are equivalent:

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