Dehn twists in Heegaard Floer homology

or, what is a Heegaard Floer homology solid torus?

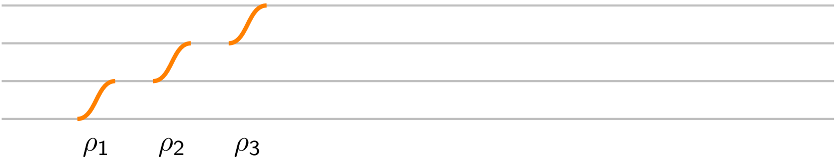
Liam Watson
University of Glasgow

www.maths.gla.ac.uk/~lwatson

CMS Ottawa
December 9, 2013

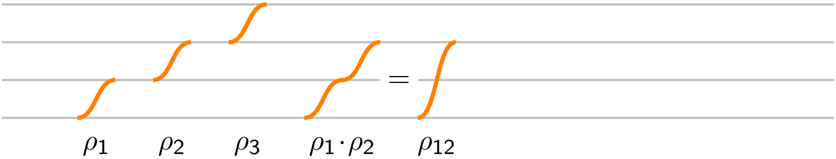
A simple algebra...

Consider an algebra $\mathcal{A}$ generated (over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$) by ρ_1, ρ_2, ρ_3:



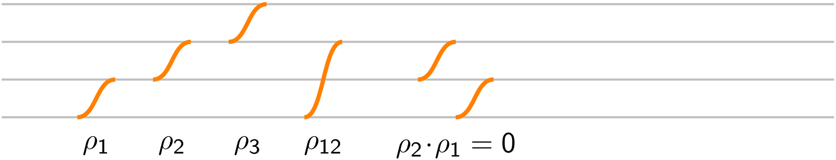
A simple algebra...

Consider an algebra $\mathcal{A}$ generated (over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$) by ρ_1, ρ_2, ρ_3:



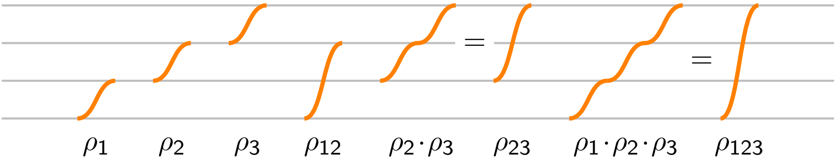
A simple algebra...

Consider an algebra $\mathcal{A}$ generated (over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$) by ρ_1, ρ_2, ρ_3:



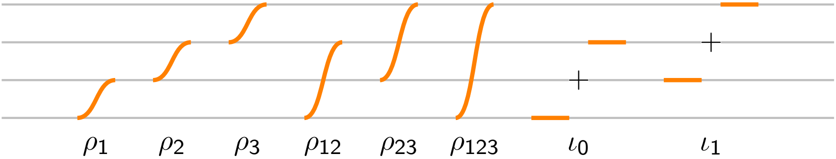
A simple algebra...

Consider an algebra $\mathcal{A}$ generated (over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$) by ρ_1, ρ_2, ρ_3:

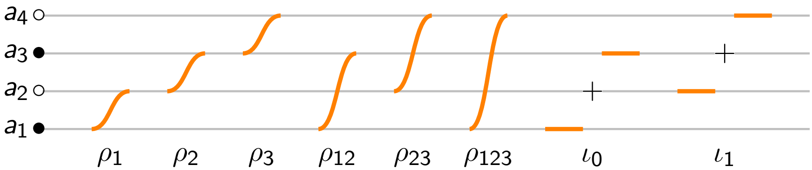


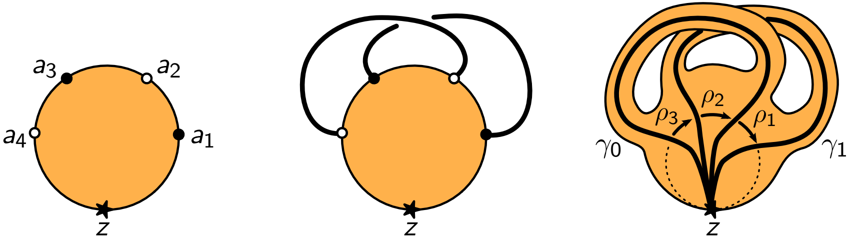
A simple algebra...

Consider an algebra $\mathcal{A}$ generated (over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$) by ρ_1, ρ_2, ρ_3:

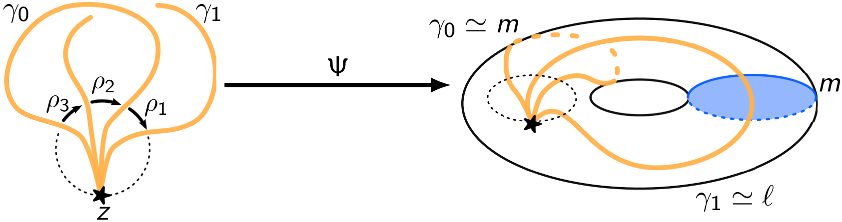


...associated with a torus.

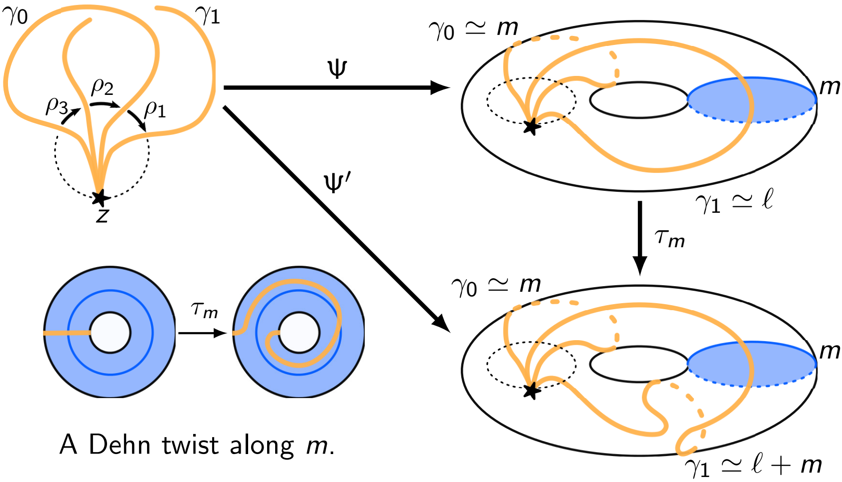




The boundary of a (relatively simple) 3-manifold



The boundary of a (relatively simple) 3-manifold



A Dehn twist along m.

Bordered structures

In general, a **bordered manifold** is a (closed, orientable) 3-manifold with parametrized boundary.

Bordered structures

In general, a **bordered manifold** is a (closed, orientable) 3-manifold with parametrized boundary.

This talk will consider the case where the boundary is a torus.

In this setting, a **bordered manifold** is an ordered triple (M, γ_0, γ_1) where γ_0, γ_1 is a parametrization of the boundary torus ∂M, or, a choice of (ordered) basis elements generating

$$\langle \gamma_0, \gamma_1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(\partial M).$$

For example

$$(M, \gamma_1, \gamma_0), (M, \gamma_0, \gamma_1), (M, \gamma_0, \gamma_0 + \gamma_1), \ldots$$

differ as bordered manifolds in general.

Bordered Heegaard Floer homology

Given a bordered manifold (M, γ_0, γ_1), **bordered Heegaard Floer homology** assigns a left differential module

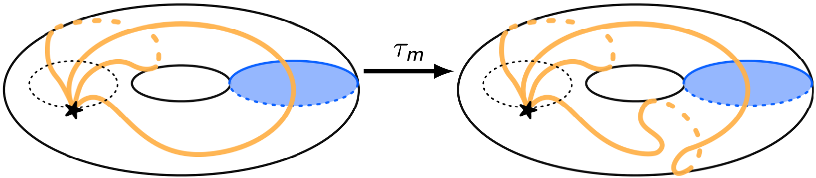
$$\widehat{\mathsf{CFD}}(M, \gamma_0, \gamma_1)$$

over the torus algebra $\mathcal{A}$.

This is an invariant of (M, γ_0, γ_1) up to homotopy.

Bordered Heegaard Floer homology was introduced by Lipshitz, Ozsváth and Thurston to study gluing (along surfaces) in Heegaard Floer homology.

The Alexander trick



Alexander trick

The Dehn twist τ_m along the meridian m extends to a homeomorphism of the solid torus.

In other words, $(D^2 \times S^1, m, \ell)$ and $(D^2 \times S^1, m, \ell + m)$ are equivalent as bordered manifolds.

Johansson's finiteness theorem

The Alexander trick characterizes the solid torus:

Theorem (Johansson)

If M is a (compact, connected, orientable, irreducible) bordered manifold for which

$$(M, \lambda, \mu) \sim (M, \lambda, \mu + \lambda)$$

as bordered manifolds, then M is a solid torus.

This follows from **Johansson's Finiteness Theorem**.

So how sensitive is this invariant to the parametrization?

Question

Does $\widehat{\mathrm{CFD}}(M, \lambda, \mu) \cong \widehat{\mathrm{CFD}}(M, \lambda, \mu + \lambda)$ *certify that* M *is a solid torus?*

So how sensitive is this invariant to the parametrization?

Question

Does $\widehat{\mathrm{CFD}}(M, \lambda, \mu) \cong \widehat{\mathrm{CFD}}(M, \lambda, \mu + \lambda)$ *certify that* M *is a solid torus?*

Definition

A Heegaard Floer homology solid torus is a bordered manifold (M, λ, μ) for which

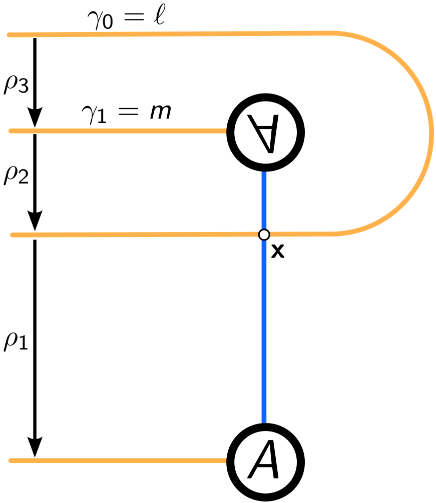
$$\widehat{\mathrm{CFD}}(M, \lambda, \mu) \cong \widehat{\mathrm{CFD}}(M, \lambda, \mu + \lambda)$$

Fine print: $H_1(M; \mathbb{Q}) = \mathbb{Q}$ and $[\lambda] \in H_1(M; \mathbb{Z})$ has finite order.

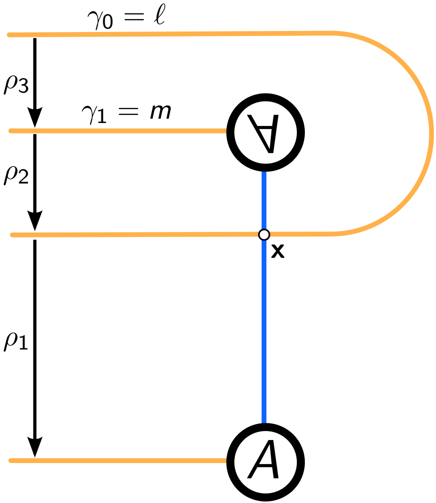
Theorem (W.)

There are infinite families of Heegaard Floer homology solid tori.

The solid torus, revisited



The solid torus, revisited

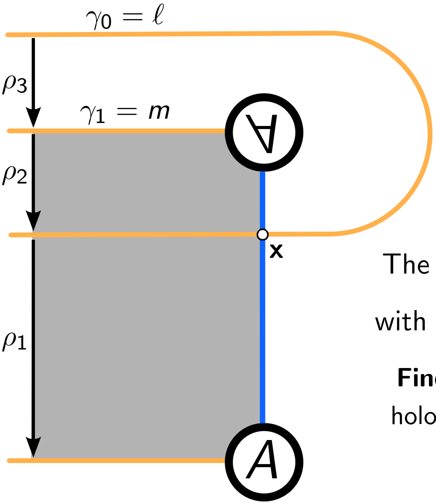


As a vector space (over $\mathbb{F}$),

$$\widehat{\mathrm{CFD}}(D^2 \times S^1, \ell, m))$$

is generated by **x**.

The solid torus, revisited

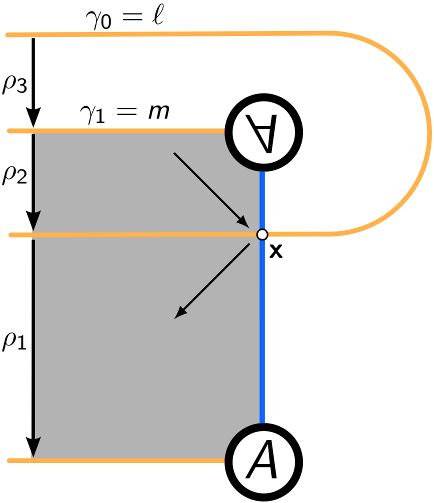


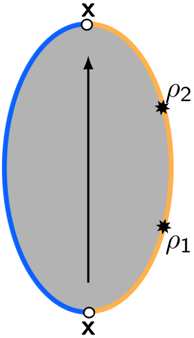
The differential counts regions in Σ

with prescribed boundary conditions.

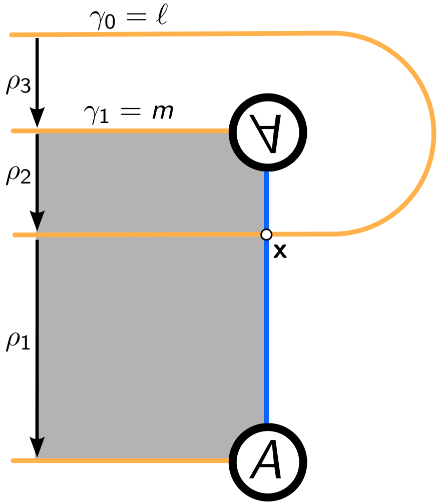
Fine print: These are projections of holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$.

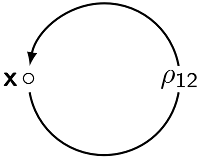
The solid torus, revisited





The solid torus, revisited





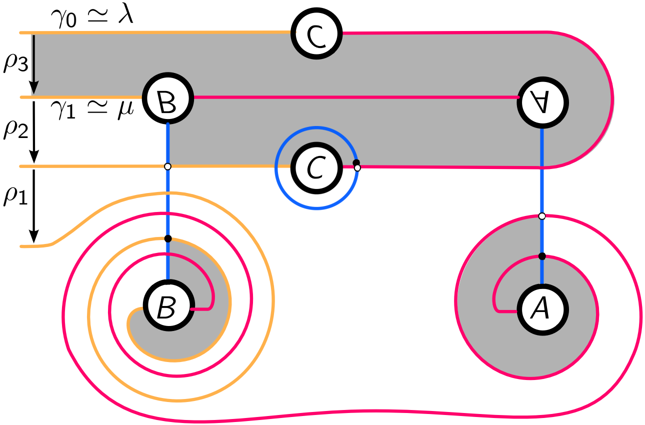
That is,

$$\widehat{\mathrm{CFD}}(D^2 \times S^1, \ell, m)$$

is generated by $\mathbf{x}$ and has differential

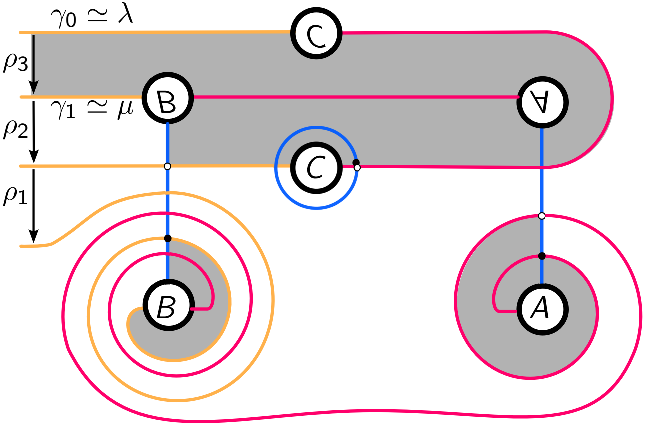
$$\partial(\mathbf{x}) = \rho_{12} \cdot \mathbf{x}.$$

More structure: Bimodules over $\mathcal{A}$



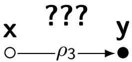
Goal: Compute $\mathbf{D}_2 = \widehat{\mathrm{CFD}}(M_2, \lambda, \mu)$

More structure: Bimodules over $\mathcal{A}$



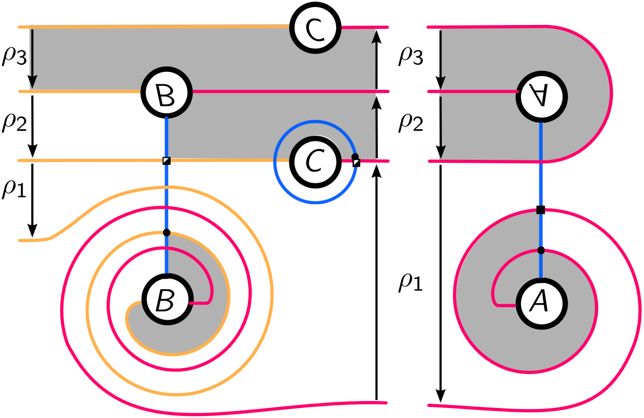
When does a region contribute to the differential?

For example,



Goal: Compute $\mathbf{D}_2 = \widehat{\mathrm{CFD}}(M_2, \lambda, \mu)$

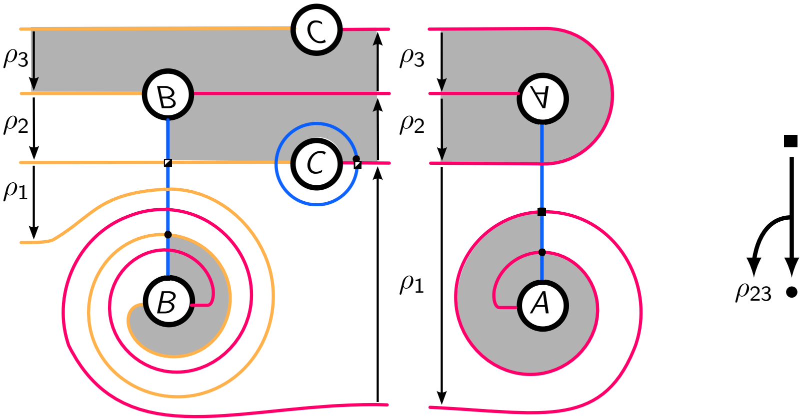
More structure: Bimodules over $\mathcal{A}$



$\mathbf{C}_2^\star = \widehat{\mathrm{CFDA}}(C_2, \Psi_L, \Psi_R)$

$\mathbf{U}_2$

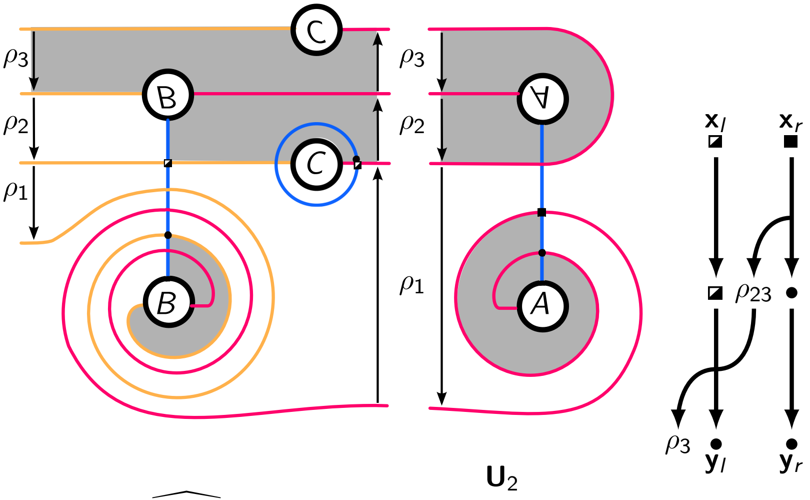
More structure: Bimodules over $\mathcal{A}$



$\mathbf{U}_2$

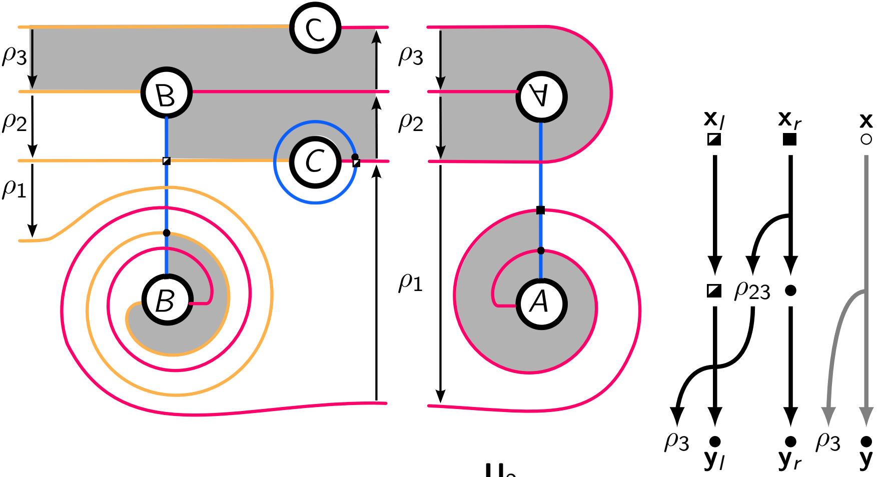
$$\mathbf{C}_2^\star = \widehat{\mathrm{CFDA}}(C_2, \Psi_L, \Psi_R)$$

More structure: Bimodules over $\mathcal{A}$



$$\mathbf{C}_2^\star = \widehat{\mathrm{CFDA}}(C_2, \Psi_L, \Psi_R)$$

More structure: Bimodules over $\mathcal{A}$



$\mathbf{C}_2^\star = \widehat{\mathrm{CFDA}}(C_2, \Psi_L, \Psi_R)$

$\mathbf{U}_2$

The paring theorem of Lipshitz, Ozsváth and Thurston

A **type D** module: $\mathbf{U}_2 = \widehat{\mathrm{CFD}}(S^2 \times S^1, \ell, m + 2\ell)$

A **type DA** bimodule: $\mathbf{C}_2^\star = \widehat{\mathrm{CFDA}}(C_2, \Psi_L, \Psi_R)$

Gives rise to a differential module generated (as a vector space) by

$$\mathbf{x} = \mathbf{x}_l \otimes_{\mathcal{I}} \mathbf{x}_r$$

with differential of the form

$$\partial(\mathbf{x}) = m_2(\mathbf{x}_l \otimes_{\mathcal{I}} \delta^1(\mathbf{x}_r)) = \rho_3 \cdot \mathbf{y}.$$

The paring theorem of Lipshitz, Ozsváth and Thurston

A **type D** module: $\mathbf{U}_2 = \widehat{\mathrm{CFD}}(S^2 \times S^1, \ell, m + 2\ell)$

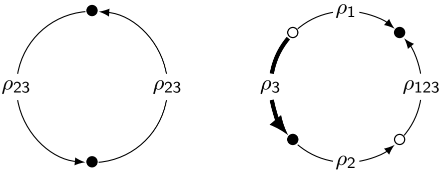
A **type DA** bimodule: $\mathbf{C}_2^\star = \widehat{\mathrm{CFDA}}(C_2, \Psi_L, \Psi_R)$

Gives rise to a differential module generated (as a vector space) by

$$\mathbf{x} = \mathbf{x}_l \otimes_{\mathcal{I}} \mathbf{x}_r$$

with differential of the form

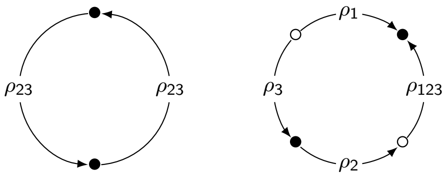
$$\partial(\mathbf{x}) = m_2(\mathbf{x}_l \otimes_{\mathcal{I}} \delta^1(\mathbf{x}_r)) = \rho_3 \cdot \mathbf{y}.$$



The paring theorem of Lipshitz, Ozsváth and Thurston

A **type D** module: $\mathbf{U}_2 = \widehat{\mathrm{CFD}}(S^2 \times S^1, \ell, m + 2\ell)$

A **type DA** bimodule: $\mathbf{C}_2^\star = \widehat{\mathrm{CFDA}}(C_2, \Psi_L, \Psi_R)$



So we get a new differential module $\mathbf{D}_2 = \mathbf{C}_2^\star \boxtimes \mathbf{U}_2$ from

$$\mathbf{x}^i = \mathbf{x}_l^i \otimes_{\mathcal{I}} \mathbf{x}_r^i \qquad \partial(\mathbf{x}^i) = \sum_{n \geq 1} m_{n+1}(\mathbf{x}_l^i \otimes_{\mathcal{I}} \delta^n(\mathbf{x}_r^i))$$

Building Heegaard Floer homology solid tori

A **type D** module: $\mathbf{U}_n = \widehat{\mathrm{CFD}}(S^2 \times S^1, \ell, n + 2\ell)$

A **type DA** bimodule: $\mathbf{C}_n^\star = \widehat{\mathrm{CFDA}}(C_n, \Psi_L, \Psi_R)$

A **type DA** bimodule: $\mathbf{C}_n = \widehat{\mathrm{CFDA}}(C_n, \Psi_R, \Psi_L)$
$= \widehat{\mathrm{CFAD}}(C_n, \Psi_L, \Psi_R)$

Building Heegaard Floer homology solid tori

A **type D** module: $\mathbf{U}_n = \widehat{\mathrm{CFD}}(S^2 \times S^1, \ell, n + 2\ell)$

A **type DA** bimodule: $\mathbf{C}_n^\star = \widehat{\mathrm{CFDA}}(C_n, \Psi_L, \Psi_R)$

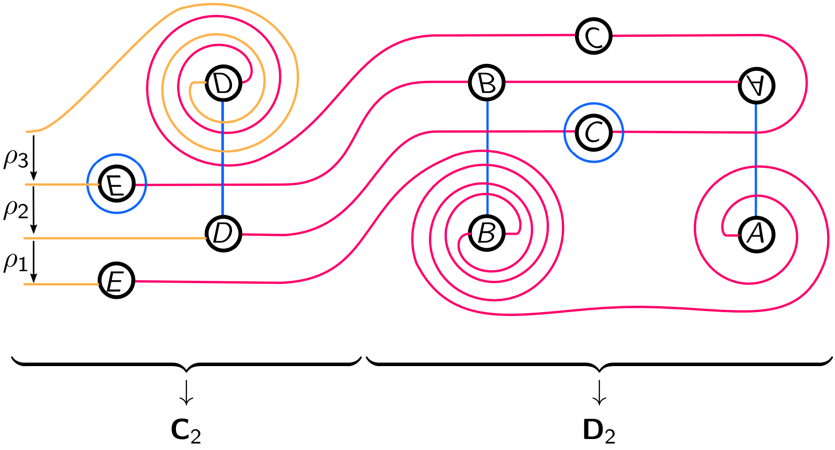
A **type DA** bimodule: $\mathbf{C}_n = \widehat{\mathrm{CFDA}}(C_n, \Psi_R, \Psi_L)$
$= \widehat{\mathrm{CFAD}}(C_n, \Psi_L, \Psi_R)$

Let $\mathbf{D}_n \cong \widehat{\mathrm{CFD}}(M_n, \lambda, \mu)$ be the differential module $\mathbf{C}_n^\star \boxtimes \mathbf{U}_n$.

Computation

$$\mathbf{C}_m \boxtimes \mathbf{D}_n \cong \underbrace{\mathbf{D}_n \oplus \cdots \oplus \mathbf{D}_n}_{m}$$

Key observation: $\mathbf{C}_1$ alters μ by a Dehn twist along λ.



D
D
E
E
B
B
C
C
A
A
ρ_3
ρ_2
ρ_1
$\mathbf{C}_2$
$\mathbf{D}_2$