KNOTS, TANGLES AND BRAID ACTIONS

by

LIAM THOMAS WATSON

B.Sc. The University of British Columbia

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

Department of Mathematics

We accept this thesis as conforming
to the required standard

. . . .

. . . .

THE UNIVERSITY OF BRITISH COLUMBIA

October 2004

Abstract

Recent work of Eliahou, Kauffmann and Thistlethwaite suggests the use of braid actions to alter a link diagram without changing the Jones polynomial. This technique produces non-trivial links (of two or more components) having the same Jones polynomial as the unlink. In this paper, examples of distinct knots that can not be distinguished by the Jones polynomial are constructed by way of braid actions. Moreover, it is shown in general that pairs of knots obtained in this way are not Conway mutants, hence this technique provides new perspective on the Jones polynomial, with a view to an important (and unanswered) question: Does the Jones polynomial detect the unknot?

Table of Contents

List of Figures

Acknowledgement

It's a strange process, writing down the details to something you've been working on for some time. And while it is a particularly solitary experience, the final product is not accomplished on you're own. To this end, there are many people that should be recognised as part of this thesis.

First, I would like to thank my supervisor Dale Rolfsen for ongoing patience, guidance and instruction. It has been a privilege and a pleasure to learn from Dale throughout my undergraduate and masters degrees, and I am grateful for his generous support, both intellectual and financial. I have learned so much.

Of course, I am indebted to all of my teachers as this work comprises much of what I have learned so far. However, I would like to single out Bill Casselman, as the figures required for this thesis could not have been produced without his guidance.

To my parents, Peter and Katherine Watson, family and friends, I am so fortunate to have a support network that allows me to pursue mathematics. In particular, I would like to thank Erin Despard for continued encouragement, support and perspective, everyday.

Chapter 1
Introduction

The study of knots begins with a straight-forward question: Can we distinguish between two closed loops, embedded in three dimensions? This leads naturally to a more general question of links, that is, the ability to distinguish between two systems of embedded closed loops. Early work by Alexander [1, 2], Artin [3, 4], Markov [24] and Reidemeister [29] made inroads into the subject, developing the first knot and link invariants, as well as the combinatorial and algebraic languages with which to approach the subject. The subtle relationship between the combinatorial and algebraic descriptions continue to set the stage for the study of knots and links.

With the discovery of the Jones polynomial [15] in 1985, along with a two variable generalization [12] shortly thereafter, the study of knots was given new focus. These new polynomial invariants could be viewed as combinatorial objects, derived directly from a diagram of the knot, or as algebraic objects, resulting from representations of the braid group. However, although the new polynomials were able to distinguish between knots that had previously caused difficulties, they led to new questions in the study of knots that have yet to be answered.

In particular, we are led to the phenomenon of distinct knots having the same Jones polynomial. There are many examples of families of knots that share common Jones polynomials. Such examples have given way to a range of tools to describe this occurrence [30, 31]. In particular, it is unknown if there is a non-trivial knot that has trivial Jones polynomial. This question motivates the understanding of knots that cannot be distinguished by the Jones polynomial, as well as the development of examples of such along with tools to explain the phenomenon. The prototypical method for producing two knots having the same Jones polynomial is known as Conway mutation. However, it is well known that this method will not alter an unknot to produce a non-trivial knot.

Recent work of Eliahou, Kauffman and Thistlethwaite [9] suggests the use of braid group actions in the study links having the same Jones polynomial. Revisiting earlier work of Kanenobu [18], new families of knots are described in this work. Once again, there is a subtle relationship between the combinatorics and the algebra associated with such examples. As a result, the study of knots obtained through braid actions can be restated in terms of fixed points of an associated group action.

The study of this braid action certainly merits attention, as the work of Eliahou, Kauffman and Thistlethwaite [9] explores Thistletwaite's discovery [33] of links having the trivial Jones polynomial, settling the question for links having more than one component. As a result, only the case of knots is left unanswered as of this writing.

This thesis is a study of families of knots sharing a common Jones polynomial. In chapter 1 the classical definitions and results of knot theory are briefly reviewed, developing the necessary background for the definitions of the Jones, Alexander and HOMFLY polynomials in chapter 2. Then, in chapter 3, the linear theory of tangles (due to Conway [8]) is carefully reviewed. Making use of this linear structure, we define a new form of mutation by way of an action of the braid group on the set of tangles.

The main results of this work are contained in chapter 5. We produce examples of distinct knots that share a common Jones polynomial, and develop a generalization of knots due to Kanenobu [18]. Moreover, it is shown (theorem 5.3) that knots constructed in this way are not related by Conway mutation. We conclude by restating the results of Eliahou, Kauffman and Thistlethwaite [9] in light of this action of the braid group, giving examples of non-trivial links having trivial Jones polynomial in chapter 6.

Chapter 2

Knots, Links and Braids

2.1 Knots and Links

A *knot* K is a smooth or piecewise linear embedding of a closed curve in a 3-dimensional manifold. Usually, the manifold of choice is either $\mathbb{R}^3$ or $\mathbb{S}^3$, so that the knot K may be denoted

$$\mathbb{S}^1 \hookrightarrow \mathbb{R}^3 \subset \mathbb{S}^3.$$

While it is important to remember that we are dealing with curves in 3-dimensions, it is difficult to work with such objects. As a result, we deal primarily with a projection of a knot to a 2-dimensional plane called a *knot diagram*. In this way a knot may be represented on the page as in figure 2.1.

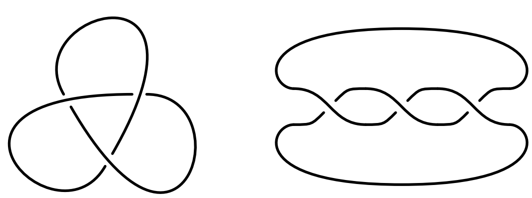
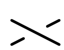


Figure 2.1: Diagrams of the Trefoil Knot

In such a diagram the  indicates that the one section of the knot (the broken line) has passed behind another (the solid line) to form a *crossing*. In general there will not be any distinction made between the knot K and a diagram representing it. That is, we allow a given diagram to represent a knot and denote the diagram by K also. It should be pointed out, however, that there are many diagrams for any given knot. Indeed, K and K' are *equivalent* knots (denoted $K \sim K'$) if they are related by isotopy in $\mathbb{S}^3$. Therefore, the diagrams for K and K' may be very different.

An n-component *link* is a collection of knots. That is, a link is a disjoint union of embedded circles

$$\coprod_{i=1}^{n} \mathbb{S}_i^1 \hookrightarrow \mathbb{R}^3 \subset \mathbb{S}^3$$

where each

$$\mathbb{S}_i^1 \hookrightarrow \mathbb{R}^3 \subset \mathbb{S}^3$$

is a knot. Of course, a 1-component link is simply a knot, and a non-trivial link can have individual components that are unknotted.

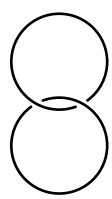
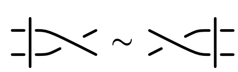


Figure 2.2: The Hopf Link

To study links by way of diagrams, it is crucial to be able to alter a link diagram in a way that reflects changes in the link resulting from isotopy in $\mathbb{S}^3$. To this end, we introduce the *Reidemeister Moves* defined in [28, 29].

(R_1)

(R_2)



(R_3)

In each of the three moves, it is understood that the diagram is unchanged outside a small disk inside which the move occurs.

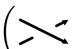
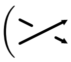
Theorem (Reidemeister). *Two link diagrams represent the same link iff the diagrams are related by planar isotopy, and the Reidemeister moves.*

Assigning an orientation to each component of a link L gives rise to the *oriented* link $\vec{L}$.

Definition 2.1. *Let $\mathcal{C}$ be the set of crossings of a diagram L. The* writhe *of an orientation $\vec{L}$ is obtained taking a sum over all crossings $\mathcal{C}$*

$$w(\vec{L}) = \sum_{c \in \mathcal{C}} w(c)$$

where $w(c) = \pm 1$ is determined by a right hand rule as in

$w\Big($  $\Big) = 1$ *and* $w\Big($  $\Big) = -1.$

While writhe is not a link invariant, it does give rise to the following definition.

Definition 2.2. *For components L_1 and L_2 of L let $\mathcal{C}' \subset \mathcal{C}$ be the set of crossings of L formed by the interaction of L_1 and L_2. The* linking number *of L_1 and L_2 is given by*

$$lk(L_1, L_2) = \sum_{c \in \mathcal{C}'} \frac{w(c)}{2}.$$

The linking number is a link invariant. Note that, for the Hopf link of figure 2.2, there are two distinct orientations. One orientation has linking number 1, the other linking number -1 and hence there are two distinct oriented Hopf links.

2.2 Braids

There are many equivalent definitions of braids (see [6], [10], [27]). In this setting it is natural to start from a geometric point of view.

Let $E \subset \mathbb{R}^3$ denote the yz-plane and let E' denote its image shifted by 1 in the x direction. Consider the the collection of points

$$\mathcal{P} = \{1, \ldots, n\} = \{(0,0,1), \ldots, (0,0,n)\} \subset E$$

and denote by

$$\mathcal{P}' = \{(1,0,1), \ldots, (1,0,n)\}$$

the image of $\mathcal{P}$ in E'.

Definition 2.3. *A (n-strand)* braid *is a collection of embedded arcs (or strands)*

$$\alpha_i : [0,1] \hookrightarrow [0,1] \times E \subset \mathbb{R}^3$$

such that

(a) $\alpha_i(0) = i \in \mathcal{P}$

(b) $\alpha_i(1) \in \mathcal{P}'$

(c) $\alpha_i \cap \alpha_j = \emptyset$ *as embedded arcs for* $i \neq j$.

(d) α_i *is monotone increasing in the* x *direction.*

As with knots, it will be convenient to consider the diagram of a braid by projecting to the xy-plane. Also, we may consider equivalence of braids via isotopy (through braids), although we will confuse the notion of a braid and its equivalence class.

In [3, 4] Artin showed that there is a well defined group structure for braids. The identity braid is represented by setting each arc to a constant map $\alpha_i(x) = (x, 0, i)$ so that each strand is a straight line. Multiplication of braids is defined by concatenation, so that inverses are constructed by reflecting in the xz-plane.

The n-strand braid group has presentation

$$B_n = \left\langle \sigma_1 \ldots \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \end{array} \right\rangle$$

where the generators correspond to a crossing formed between the i and $i+1$ strand as in figure 2.3.

Just as group elements are formed by words in the generators, a braid diagram for a given element can be constructed by concatenation of braids of the form shown in figure 2.3.

If E and E' of an n-strand braid are identified so that $\mathcal{P} = \mathcal{P}'$ pointwise, the result is a collection of embeddings of $\mathbb{S}^1$ in $\mathbb{R}^3$ and we obtain a link.

Given any braid β we can form a link $\bar{\beta}$ by taking the closure in this way. It is a theorem of Alexander [1] that every link arises as the closure of some braid. Given a link diagram L, it is always possible to construct a braid β such that $\bar{\beta} = L$. Two such constructions (there are many) are due to Morton [26] and Vogel [34].

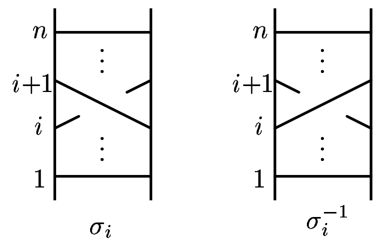


Figure 2.3: The braid generator σ_i and its inverse

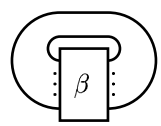


Figure 2.4: The link $\bar{\beta}$ formed from the closure of β.

Now it should be noted that the group operation

$$\sigma_i \sigma_i^{-1} = 1 = \sigma_i^{-1} \sigma_i$$

corresponds exactly to the Reidemeister move R_2, while the group relation

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$$

corresponds to the Reidemeister move R_3. This suggests the possibility of studying equivalence of links through braid representatives. To this end we define the *Markov moves*. Suppose $\beta \in B_n$ and write $\beta = (\beta, n)$. Then

$$(\beta_1 \beta_2, n) \sim_{\mathrm{M}} (\beta_2 \beta_1, n) \qquad (\mathrm{M}_1)$$

$$(\beta, n) \sim_{\mathrm{M}} (\beta \sigma_n^{\pm 1}, n+1) \qquad (\mathrm{M}_2)$$

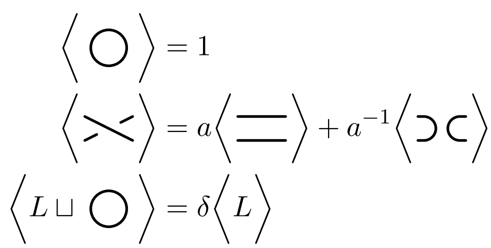
where $\sim_{\mathrm{M}}$ denotes Markov equivalence. The following theorem, due to Markov [24], is proved in detail in [6].

Theorem (Markov). *Two links $\bar{\beta}_1$ and $\bar{\beta}_2$ are equivalent iff $\beta_1 \sim_M \beta_2$.*

Chapter 3
Polynomials

3.1 The Jones Polynomial

Define the *Kauffman bracket* $\langle L \rangle$ of a link diagram L recursively by the axioms



(3.1)

(3.2)

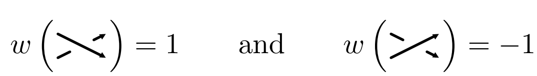
(3.3)

where a is a formal variable and

$$\delta = -a^{-2} - a^2$$

so that $\langle L \rangle$ is an element of the (Laurent) polynomial ring $\mathbb{Z}[a, a^{-1}]$. In some cases a is specified as a non-zero complex number, in which case $\langle L \rangle \in \mathbb{C}$.

The Kauffman bracket is invariant under the Reidemeister moves R_2 and R_3. To get invariance under R_1, we recall definition 2.1 for the writhe of an orientation $\vec{L}$ of the diagram L. The writhe of a crossing is ± 1 and is determined by a right hand rule. That is



so that $w(\vec{L}) \in \mathbb{Z}$. Now

$$-a^{-3w(\vec{L})} \langle L \rangle \in \mathbb{Z}[a, a^{-1}]$$

is invariant under R_1 and gives rise to an invariant of oriented links.

Definition 3.1. *The* Jones Polynomial [15, 19] *is given by*

$$V_{\vec{L}}(t) = -a^{-3w(\vec{L})}\langle L\rangle\Big|_{a=t^{-\frac{1}{4}}}$$

where t is a commuting variable.

Note that it will often be convenient to work with $t = a^{-4}$, and the polynomial obtained through this substitution will be referred to as the Jones Polynomial also.

As we shall see, there are many examples of distinct links having the same Jones Polynomial. However, the following is still unknown:

Question 3.2. *For a knot K, does $V_{\vec{K}}(t) = 1$ imply that $K \sim \bigcirc$?*

3.2 The Alexander Polynomial

For any knot K, let F be an orientable surface such that $\partial F = K$. Such a surface always exists [32], and is called a *Seifert surface* for the link K. The homology of such a surface is given by

$$H_1(F, \mathbb{Z}) = \bigoplus_{2g} \mathbb{Z}$$

where g is the genus of the surface F. Let $\{a_i\}$ be a set of generators for $H_1(F, \mathbb{Z})$ where $i \in \{1, \ldots, 2g\}$.

Let

$$D^2 = \{z \in \mathbb{C} : |z| < 1\}$$

and consider a tubular neighborhood $\mathcal{N}(K) \cong K \times D^2$ of the link K. That is, an embedding

$$\mathbb{S}^1 \times D^2 \hookrightarrow \mathbb{S}^3$$

such that K is the restriction to $\mathbb{S}^1 \times \{0\}$.

Now consider the surface F in the complement $X = \mathbb{S}^3 \smallsetminus \mathcal{N}(K)$. Here F is being confused with its image in the compliment X, by abuse. For a regular neighborhood

$$F \times [1, 1] \subset \mathbb{S}^3$$

there are natural inclusions

$$i^{\pm} : F \hookrightarrow F \times \{\pm 1\}$$

where $F = F \times \{0\}$ is the Seifert surface in X. Therefore a cycle $x \in H_1(F, \mathbb{Z})$ gives rise to a cycle $x^{\pm} = i^{\pm}_{\star}(x) \in H_1(X, \mathbb{Z})$.

Definition 3.3. *The* Seifert Form *is the bilinear form*

$$\begin{aligned} v : H_1(F,\mathbb{Z}) \times H_1(F,\mathbb{Z}) &\to \mathbb{Z} \\ (x, y) &\mapsto lk(x, y^+) \end{aligned}$$

and it is represented by the Seifert Matrix

$$V = \left(lk(a_i, a_j^+) \right)$$

where $y^+ = i^+_{\star}(y)$.

The aim is to construct $\widetilde{X}$, the *infinite cyclic cover* [25, 32] of the knot complement $X = \mathbb{S}^3 \smallsetminus \mathcal{N}(K)$. To do this, start with a countable collection $\{X_i\}_{i \in \mathbb{Z}}$ of

$$X_i = X \smallsetminus (F \times (-\epsilon, \epsilon))$$

for some small $\epsilon \in (0, 1)$. The boundary of this space contains two identical copies of F denoted by

$$F^{\pm} = F \times \{\pm\epsilon\},$$

and the infinite cyclic cover of X is defined

$$\widetilde{X} = \bigcup_{i \in \mathbb{Z}} X_i \Big/ F_i^+ \sim F_{i+1}^-$$

by identifying $F_i^+ \subset \partial X_i$ with $F_{i+1}^- \subset \partial X_{i+1}$ for each $i \in \mathbb{Z}$.

The space obtained corresponds to the short exact sequence

$$1 \longrightarrow \pi_1 \widetilde{X} \longrightarrow \pi_1 X \longrightarrow H_1(X, \mathbb{Z}) \longrightarrow 0$$

$$\alpha \longmapsto lk\left(\alpha, \vec{K}\right)$$

so that the infinite cyclic group $H_1(X, \mathbb{Z}) = \langle t \rangle$ gives the covering translations of $\widetilde{X} \searrow X$. Now $H_1(\tilde{X}, \mathbb{Z})$, although typically not finitely generated as an abelian group, is finitely generated as a $\mathbb{Z}[t, t^{-1}]$-module by the $\{a_i\}$. The variable t corresponds to the $\langle t \rangle$-action taking X_i to X_{i+1}.

Definition 3.4. $H_1(\tilde{X}, \mathbb{Z})$ *is called the* Alexander module *and has module presentation* $V - tV^\top$*, where* V *is the Seifert matrix. This is a knot invariant.*

This gives rise to another polynomial invariant due to Alexander [2].

Definition 3.5. *The Laurent polynomial*

$$\Delta_K(t) \doteq \det(V - tV^\top) \in \mathbb{Z}[t, t^{-1}]$$

is an invariant of the knot K *called the* Alexander polynomial. *It is defined up to multiplication by a unit* $\pm t^{\pm n}$ *(indicated by* $\doteq$*).*

This knot invariant is of particular interest due to this topological construction.

Question 3.6. *Is there a similar topological interpretation for the Jones polynomial?*

On the other hand, it is easy to generate knots K such that $\Delta_K(t) = 1$ (see for example, [32]).

Theorem 3.7. *For any knot* K*,* $\Delta_K(t) \doteq \Delta_K(t^{-1})$.

PROOF. Given the $n \times n$ Seifert matrix V,

$$\begin{aligned}
\Delta_K &\doteq \det(V - tV^\top) \\
&= \det(V^\top - tV) \\
&= (-t)^n \det(V - t^{-1}V^\top) \\
&\doteq \Delta_K(t^{-1}).
\end{aligned}$$

□

Theorem 3.8. *For any knot* K*,* $\Delta_K(1) = \pm 1$.

PROOF. Setting $t = 1$ and using the standard (symplectic) basis for $H_1(F, \mathbb{Z})$

gives

$$\begin{aligned} V - V^\top &= \Big(lk(a_i, a_j^+) - lk(a_j, a_i^+) \Big) \\ &= \Big(lk(a_i, a_j^+) - lk(a_j^-, a_i) \Big) \\ &= \Big(lk(a_i, a_j^+) - lk(a_i, a_j^-) \Big) \\ &= \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix} \end{aligned}$$

therefore

$$\Delta_K(1) = \pm \det(V - V^\top) = \pm 1.$$

□

Corollary 3.9. *For any knot K*

$$\Delta_K(t) \doteq c_0 + c_1(t^{-1} + t^1) + c_2(t^{-1} + t^1)^2 + \cdots$$

where $c_i \in \mathbb{Z}$.

PROOF. The symmetry given by $\Delta_K(t) \doteq \Delta_K(t^{-1})$ gives rise to the form

$$\Delta_K(t) \doteq \sum_{k=0}^{m} b_k t^k$$

where $c_{m-r} = \pm c_r$, and the same choice of sign is made for each r. Now m must be even, since m odd gives rise to $\Delta_K(1)$ even, contradicting theorem 3.8. Further, if $c_{m-r} = -c_r$ then $c_{m/2} = 0$ and

$$\Delta_K(1) = \sum_{k=0}^{m} b_k = 0,$$

again contradicting theorem 3.8. Therefore $c_{m-r} = c_r$, and

$$\Delta_K(t) \doteq c_0 + c_1(t^{-1} + t^1) + c_2(t^{-1} + t^1)^2 + \cdots$$

as required. □

We will see the form given in corollary 3.9 in the next section. Together with the normalization $\Delta_K(1) = 1$, it is sometimes referred to as the *Alexander-Conway polynomial* as it has a recursive definition, originally noticed by Alexander [2] and later exploited by Conway [8].

It should be noted that there are generalizations of this construction to invariants of oriented links that have been omitted. Nevertheless, we shall see that the recursive definition of $\Delta_K(t)$ is defined for all oriented links.

3.3 The HOMFLY Polynomial

A two variable polynomial [12, 16] that restricts to each of the polynomials introduced may be defined, albeit by very different means.

The n-strand braid group B_n generates a group algebra H_n over $\mathbb{Z}[q, q^{-1}]$ which has relations

$$\begin{array}{lll} \text{(i)} & \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} \quad |i-j| > 1 \\ \text{(ii)} & \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} \quad |i-j| = 1 \\ \text{(iii)} & \sigma_i^2 = (q-1)\sigma_i + q & \forall \quad i \in \{1, \ldots, n-1\} \end{array}$$

called the *Hecke* algebra. By allowing q to take values in $\mathbb{C}$, H_n can be seen as a quotient of the group algebra $\mathbb{C}B_n$. Just as

$$\{1\} < B_2 < B_3 < B_4 < \cdots$$

we have that

$$\mathbb{Z}[q, q^{-1}] \subset H_2 \subset H_3 \subset H_4 \subset \cdots .$$

Note that for $q = 1$, the relation (iii) reduces to $\sigma_i^2 = 1$ and we obtain the relations for the symmetric group S_n.

Definition 3.10. *Sets of* positive permutation braids *may be defined recursively via*

$$\begin{array}{ll} \Sigma_0 = \{1\} & \\ \Sigma_i = \{1\} \cup \sigma_i\Sigma_{i-1} & \textit{for} \quad i > 0. \end{array}$$

A monomial $m \in H_n$ is called normal *if it has the form*

$$m = m_1 m_2 \ldots m_{n-1}$$

where $m_i \in \Sigma_i$.

The normal monomials form a basis for H_n, and it follows that

$$\dim_{\mathbb{Z}[q,q^{-1}]}(H_n) = n!.$$

Moreover, this basis allows us to present any element of H_{n+1} in the form

$$x_1 + x_2\sigma_n x_3$$

for $x_i \in H_n$. The relation (i) implies that

$$x\sigma_n = \sigma_n x$$

whenever $x \in H_{n-1}$, giving rise to the decomposition

$$H_{n+1} \cong H_n \oplus \left(H_n \otimes_{H_{n-1}} H_n\right).$$

Now we define a linear trace function

$$\begin{aligned} \mathrm{tr} : H_n &\longrightarrow \mathbb{Z}[q^{\pm 1}, z] \\ \sigma_i &\longmapsto z \end{aligned}$$

that is normalized so that $\mathrm{tr}(1) = 1$.

Theorem 3.11. $\mathrm{tr}(x_1x_2) = \mathrm{tr}(x_2x_1)$ *for* $x_i \in H_n$.

PROOF. By linearity, it suffices to show that $\mathrm{tr}(m_1m_2) = \mathrm{tr}(m_2m_1)$ for normal monomials $m_i \in H_n$. Since the theorem is clearly true for the normal monomials of H_2, we proceed by induction.

Suppose first that $m_1 = m_1'\sigma_n m_1''$ where $m_1', m_1'' \in H_n$ and $m_2 \in H_n$ (that is, m_2 contains no σ_n). Then

$$\begin{aligned} \mathrm{tr}(m_1m_2) &= \mathrm{tr}(m_1'\sigma_n m_1'' m_2) \\ &= z\,\mathrm{tr}(m_1' m_1'' m_2) \\ &= z\,\mathrm{tr}(m_2 m_1' m_1'') && \text{by induction} \\ &= \mathrm{tr}(m_2 m_1'\sigma_n m_1'') \\ &= \mathrm{tr}(m_2m_1). \end{aligned}$$

Now more generally write

$$m_1 = m_1'\sigma_n m_1'' \quad \text{and} \quad m_2 = m_2'\sigma_n m_2''$$

where $m_i', m_i'' \in H_n$. In this case we will make use of the following:

(1) $\mathrm{tr}(\mu_1\sigma_n\mu_2\sigma_n) = \mathrm{tr}(\sigma_n\mu_1\sigma_n\mu_2)$

(2) $\mathrm{tr}(\mu_1\sigma_n\mu_2\sigma_n\mu_3) = \mathrm{tr}(\mu_3\mu_1\sigma_n\mu_2\sigma_n)$

where $\mu_i \in H_n$ are in normal form so that $\mu_i = \mu_i'\sigma_{n-1}\mu_i''$ with $\mu_i', \mu_i'' \in H_{n-1}$.

(1)

$$\begin{aligned}
\mathrm{tr}(\mu_1\sigma_n\mu_2\sigma_n) &= \mathrm{tr}(\mu_1\sigma_n\mu_2'\sigma_{n-1}\mu_2''\sigma_n) \\
&= \mathrm{tr}(\mu_1\mu_2'\sigma_n\sigma_{n-1}\sigma_n\mu_2'') && \text{using (i)} \\
&= \mathrm{tr}(\mu_1\mu_2'\sigma_{n-1}\sigma_n\sigma_{n-1}\mu_2'') && \text{using (ii)} \\
&= z\,\mathrm{tr}(\mu_1\mu_2'\sigma_{n-1}^2\mu_2'') \\
&= z\,\mathrm{tr}(\mu_1\mu_2'[(q-1)\sigma_{n-1}+q]\mu_2'') && \text{using (iii)} \\
&= z(q-1)\,\mathrm{tr}(\mu_1\mu_2'\sigma_{n-1}\mu_2'') + zq\,\mathrm{tr}(\mu_1\mu_2'\mu_2'') \\
&= z(q-1)\,\mathrm{tr}(\mu_1\mu_2) + zq\,\mathrm{tr}(\mu_1'\sigma_{n-1}\mu_1''\mu_2'\mu_2'') \\
&= z(q-1)\,\mathrm{tr}(\mu_1\mu_2) + zq\,\mathrm{tr}(\mu_1'\mu_1''\mu_2'\sigma_{n-1}\mu_2'') && \text{induction} \\
&= z(q-1)\,\mathrm{tr}(\mu_1\mu_2) + zq\,\mathrm{tr}(\mu_1'\mu_1''\mu_2) \\
&= z\,\mathrm{tr}(\mu_i'[(q-1)\sigma_{n-1}+q]\mu_1''\mu_2) \\
&= z\,\mathrm{tr}(\mu_1'\sigma_{n-1}^2\mu_1''\mu_2) && \text{using (iii)} \\
&= \mathrm{tr}(\mu_1'\sigma_{n-1}\sigma_n\sigma_{n-1}\mu_1''\mu_2) \\
&= \mathrm{tr}(\mu_1'\sigma_n\sigma_{n-1}\sigma_n\mu_1''\mu_2) && \text{using (ii)} \\
&= \mathrm{tr}(\sigma_n\mu_1'\sigma_{n-1}\mu_1''\sigma_n\mu_2) && \text{using (i)} \\
&= \mathrm{tr}(\sigma_n\mu_1\sigma_n\mu_2)
\end{aligned}$$

(2)

$$\begin{aligned}
&\mathrm{tr}(\mu_1\sigma_n\mu_2\sigma_n\mu_3) \\
&\quad= \mathrm{tr}(\mu_1\mu_2\sigma_n\mu_3\sigma_n) && \text{applying (1)} \\
&\quad= \mathrm{tr}(\mu_1\mu_2\sigma_n\mu_3'\sigma_{n-1}\mu_3''\sigma_n) \\
&\quad= z\,\mathrm{tr}(\mu_1\mu_2\mu_3'\sigma_{n-1}^2\mu_3'') && \text{as above} \\
&\quad= z(q-1)\,\mathrm{tr}(\mu_1\mu_2\mu_3'\sigma_{n-1}\mu_3'') + zq\,\mathrm{tr}(\mu_1\mu_2\mu_3'\mu_3'') && \text{using (iii)} \\
&\quad= z(q-1)\,\mathrm{tr}(\mu_3'\sigma_{n-1}\mu_3''\mu_1\mu_2) + zq\,\mathrm{tr}(\mu_3'\mu_3''\mu_1\mu_2) && \text{induction} \\
&\quad= z\,\mathrm{tr}(\mu_3'\sigma_{n-1}^2\mu_3''\mu_1\mu_2) && \text{using (iii)} \\
&\quad= \mathrm{tr}(\sigma_n\mu_3'\sigma_{n-1}\mu_3''\sigma_n\mu_1\mu_2) && \text{as above} \\
&\quad= \mathrm{tr}(\sigma_n\mu_3\sigma_n\mu_1\mu_2) \\
&\quad= \mathrm{tr}(\mu_3\mu_1\sigma_n\mu_2\sigma_n) && \text{applying (1)}
\end{aligned}$$

Now the proof is complete, since

$$\begin{aligned}
\operatorname{tr}(m_1 m_2) &= \operatorname{tr}(m_1' \sigma_n m_1'' m_2' \sigma_n m_2'') \\
&= \operatorname{tr}(m_2'' m_1' \sigma_n m_1'' m_2' \sigma_n) && \text{by (2)} \\
&= \operatorname{tr}(\sigma_n m_2'' m_1' \sigma_n m_1'' m_2') && \text{by (1)} \\
&= \operatorname{tr}(m_2' \sigma_n m_2'' m_1' \sigma_n m_1'') && \text{by (2)} \\
&= \operatorname{tr}(m_2 m_1).
\end{aligned}$$

□

Now since elements of H_{n+1} are of the form $x_1 + x_2\sigma_n x_3$ where $x_i \in H_n$, the trace function may be extended from H_n to H_{n+1} by

$$\operatorname{tr}(x_1 + x_2\sigma_n x_3) = \operatorname{tr}(x_1) + z\operatorname{tr}(x_2 x_3).$$

The aim is to use the trace function to define a link invariant. In particular we would like to make use of this trace on braids in the composite

$$B_n \longrightarrow H_n \longrightarrow \mathbb{Z}[q^{\pm 1}, z].$$

To do this, we introduce a change of variables $\lambda = \frac{w}{qz}$ where

$$z = -\frac{1-q}{1-\lambda q} \quad \text{and} \quad w = -\frac{\lambda q(1-q)}{1-\lambda q}$$

so that

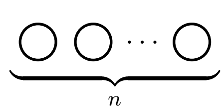
$$\lambda = \frac{1-q+z}{zq}.$$

Definition 3.12. *The* HOMFLY *polynomial is given by*

$$X_{\bar{\beta}}(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} \left(\sqrt{\lambda}\right)^e \operatorname{tr}(\beta)$$

where $\beta \in B_n$ is a monomial in H_n and $e = e(\beta)$ is the exponent sum of β (equivalently, the abelianization $B_n \to \mathbb{Z}$).

Note that the closure of the identity braid in B_n gives the n component unlink



and the HOMFLY polynomial for this link is given by

$$\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}. \tag{3.4}$$

Theorem 3.13. *Let* $\bar{\beta} = L$ *then* $X_L(q,\lambda) \in \mathbb{Z}\left[q^{\pm 1}, \left(\sqrt{\lambda}\right)^{\pm 1}\right]$ *is a link invariant.*

PROOF. By Markov's theorem, we need only check that $X_L(q,\lambda)$ is invariant under M_1 and M_2. The fact that $\mathrm{tr}(\beta_1\beta_2) = \mathrm{tr}(\beta_2\beta_1)$ from Theorem 3.11 gives invariance under M_1, so it remains to check invariance under M_2. Suppose then that $\beta \in B_n$. With the above substitution we have

$$\mathrm{tr}(\sigma_n) = -\frac{1-q}{1-\lambda q}$$

so that

$$\begin{aligned} X_{\overline{\beta\sigma_n}}(q,\lambda) &= \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n} \left(\sqrt{\lambda}\right)^{e+1} \mathrm{tr}(\beta\sigma_i) \\ &= \sqrt{\lambda}\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)\left(-\frac{1-q}{1-\lambda q}\right) X_{\bar{\beta}}(q,\lambda) \\ &= X_{\bar{\beta}}(q,\lambda). \end{aligned}$$

Further, from (iii) we can derive

$$\begin{aligned} \sigma_i^2 &= (q-1)\sigma_i + q \\ \sigma_i &= (q-1) + q\sigma_i^{-1} \\ q\sigma_i^{-1} &= \sigma_i + 1 - q \\ \sigma_i^{-1} &= q^{-1}\sigma_i + q^{-1} - 1 \end{aligned}$$

hence

$$\begin{aligned}
\operatorname{tr}(\sigma_i^{-1}) &= \operatorname{tr}(q^{-1}\sigma_i + q^{-1} - 1) \\
&= q^{-1}\operatorname{tr}(\sigma_i) + q^{-1} - 1 \\
&= q^{-1}\left(-\frac{1-q}{1-\lambda q}\right) + q^{-1} - 1 \\
&= q^{-1}\left(-\frac{1-q}{1-\lambda q} + 1\right) - 1 \\
&= q^{-1}\left(\frac{-1+q+1-\lambda q}{1-\lambda q}\right) - 1 \\
&= \frac{1-\lambda-1+\lambda q}{1-\lambda q} \\
&= -\lambda\frac{1-q}{1-\lambda q}.
\end{aligned}$$

Thus

$$\begin{aligned}
X_{\overline{\beta\sigma_n^{-1}}}(q,\lambda) &= \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^n \left(\sqrt{\lambda}\right)^{e-1} \operatorname{tr}(\beta\sigma_1^{-1}) \\
&= \frac{1}{\sqrt{\lambda}}\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)\left(-\lambda\frac{1-q}{1-\lambda q}\right) X_{\bar{\beta}}(q,\lambda) \\
&= X_{\bar{\beta}}(q,\lambda)
\end{aligned}$$

and $X_{\bar{\beta}}(q,\lambda)$ is a link invariant. □

Both the single variable polynomials may be retrieved from the HOMFLY polynomial via the substitutions

$$\begin{aligned}
V_L(t) &= X_L(t,t) \\
\Delta_L(t) &\doteq X_L\left(t,t^{-1}\right).
\end{aligned}$$

Another definition of the HOMFLY polynomial is possible. For $\beta \in B_n$ suppose that $L = \bar{\beta}$, oriented so that the generator σ_i is a positive crossing (that is, $w(\sigma_i) = 1$). Suppose that β contains some σ_i [1] and write

$$\beta = \gamma_1\sigma_i\gamma_2$$

[1] a similar construction is possible for σ_i^{-1}

for $\gamma_i \in B_n$. By applying M_1 we can define

$$L \sim L_0 = \overline{\gamma\sigma_i}$$

where $\gamma = \gamma_1\gamma_2$. Let

$$L_+ = \overline{\gamma\sigma_i^2} \quad \text{and} \quad L_- = \bar{\gamma}.$$

The relation (iii) gives

$$\gamma\sigma_i^2 = (q-1)\gamma\sigma_i + q\gamma$$

so that

$$\operatorname{tr}(\gamma\sigma_i^2) - q\operatorname{tr}(\gamma) = (q-1)\operatorname{tr}(\gamma\sigma_i).$$

Let $e = e(\gamma)$ be the exponent sum of γ. Then

$$\begin{aligned}
&\frac{1}{\sqrt{q}\sqrt{\lambda}}X_{\overline{\gamma\sigma_i^2}} - \sqrt{q}\sqrt{\lambda}X_{\bar{\gamma}} \\
=&\frac{1}{\sqrt{q}\sqrt{\lambda}}\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}\left(\sqrt{\lambda}\right)^{e+2}\operatorname{tr}(\gamma\sigma_i^2) \\
&\quad - \sqrt{q}\sqrt{\lambda}\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}\left(\sqrt{\lambda}\right)^{2}\operatorname{tr}(\gamma) \\
=&\frac{\left(\sqrt{\lambda}\right)^{e+1}}{\sqrt{q}}\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}\left(\operatorname{tr}(\gamma\sigma_i^2) - q\operatorname{tr}(\gamma)\right) \\
=&\frac{\left(\sqrt{\lambda}\right)^{e+1}}{\sqrt{q}}\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}\left((q-1)\operatorname{tr}(\gamma\sigma_i)\right) \\
=&\left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}\left(\sqrt{\lambda}\right)^{e+1}\operatorname{tr}(\gamma\sigma_1) \\
=&\left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)X_{\overline{\gamma\sigma_1}}
\end{aligned}$$

and this shows that $X_L(q,\lambda)$ satisfies the *skein relation*

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}X_{L_+}(q,\lambda) - \sqrt{q}\sqrt{\lambda}X_{L_-}(q,\lambda) = \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)X_{L_0}(q,\lambda).$$

By introducing the substitution

$$t = \sqrt{q}\sqrt{\lambda} \quad \text{and} \quad x = \sqrt{q} - \frac{1}{\sqrt{q}}$$

we can define

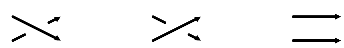
$$P_L(t,x) = X_L(q,\lambda)$$

where $P_L(t,x) \in \mathbb{Z}[t^{\pm 1}, x^{\pm 1}]$ is computed recursively from the axioms

$$P_{\bigcirc}(t,x) = 1 \tag{3.5}$$

$$t^{-1}P_{L_+}(t,x) - tP_{L_-}(t,x) = xP_{L_0}(t,x). \tag{3.6}$$

In this setting, L_+, L_- and L_0 are diagrams that are identical except for in a small region where they differ as in



L_+ L_- L_0

By a simple application of (3.6), the polynomial of the n component unlink is

$$\left(\frac{t^{-1}-t}{x}\right)^{n-1} \tag{3.7}$$

in the skein definition of the HOMFLY polynomial. This agrees with (3.4) under the substitutions

$$t = \sqrt{q}\sqrt{\lambda} \quad \text{and} \quad x = \sqrt{q} - \frac{1}{\sqrt{q}}.$$

As indicated earlier, both the Jones polynomial and the Alexander polynomial may be computed recursively as they each satisfy a skein relation by specifying

$$V_L(t) = P_L\left(t, \sqrt{t} - \frac{1}{\sqrt{t}}\right)$$
$$\Delta_L(t) \doteq P_L\left(1, \sqrt{t} - \frac{1}{\sqrt{t}}\right).$$

Chapter 4

Tangles and Linear Maps

4.1 Conway Tangles

In the recursive computation of the Kauffman bracket of a link, the order in which the crossings are reduced is immaterial. In many cases it will be convenient to group crossings together in the course of computation. From Conway's point of view [8], such groupings or *tangles* form the building blocks of knots and links. In addition, this point of view will allow us to take advantage of the well-developed tools of linear algebra.

Definition 4.1. *Given a link L in $\mathbb{S}^3$ consider a 3-ball $B^3 \subset \mathbb{S}^3$ such that ∂B^3 intersects L in exactly 4 points. The intersection $B^3 \cap L$ is called a* <u>Conway</u> tangle *(or simply, a* tangle*) denoted by T. The exterior of the tangle $\overline{\mathbb{S}^3 \smallsetminus B^3} \cap L$ is called an* external wiring*, denoted by $L \smallsetminus T$.*

Note that, as $\overline{\mathbb{S}^3 \smallsetminus B^3}$ is a ball, the external wiring $L \smallsetminus T$ is a tangle also.

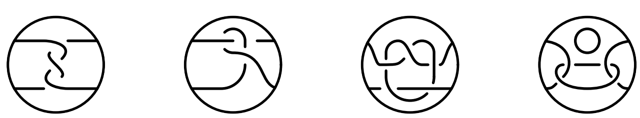
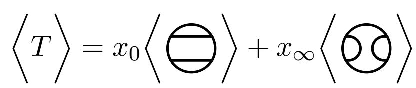


Figure 4.1: Some diagrams of Conway tangles

A tangle, as a subset of a link, may be considered up to equivalence under isotopy. When a diagram of the link L is considered, a tangle may be represented by a disk in the projection plane, with boundary intersecting the link in 4 points. Equivalence of tangle diagrams then, is given by the Reidemeister moves, where the four boundary points are fixed.

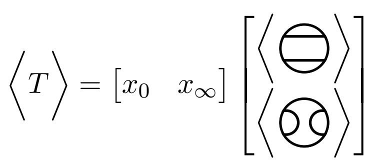
Further, the Kauffman bracket of a tangle T may be computed by way of the axioms (3.2) and (3.3). Thus the the Kauffman bracket of any tangle may be written in terms of tangles having no crossings or closed loops. There are only two such tangles, and they are denoted by

These tangles are fundamental in the sense that they form a basis for presenting the bracket of a given tangle T. That is



where $x_0, x_\infty \in \mathbb{Z}[a, a^{-1}]$.

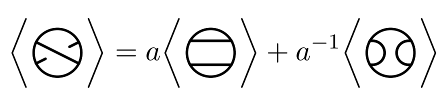
Definition 4.2. *Let T be a Conway tangle and*



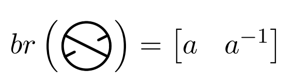
where $x_0, x_\infty \in \mathbb{Z}[a, a^{-1}]$. The bracket vector *of T is denoted*

$$br(T) = \begin{bmatrix} x_0 & x_\infty \end{bmatrix}.$$

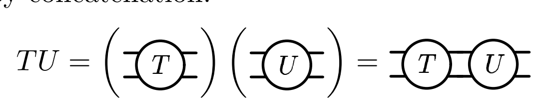
In this way, the Kauffman bracket divides Conway tangles into equivalence classes completely determined by $br(T)$. For example,



and



We can define a product for tangles that is similar to multiplication in the braid group. Given Conway tangles T and U the product TU is a Conway tangle obtained by concatenation:



Notice that when $T \in B_2$ this is exactly braid multiplication.

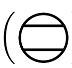
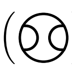
Definition 4.3. *The* Kauffman bracket skein module $\mathcal{S}^{br}$ *is the* $\mathbb{Z}[a, a^{-1}]$*-module generated by isotopy classes of Conway tangles, modulo equivalence given by axioms (3.2) and (3.3) defining the Kauffman bracket.*

The tangles $\{0, \infty\}$ provide a module basis for $\mathcal{S}^{br}$ so that the elements $T \in \mathcal{S}^{br}$ may be represented by $br(T)$.

Suppose the tangle T is contained in some link L. Then writing $L = L(T)$ and considering $T \in \mathcal{S}^{br}$ gives rise to a $\mathbb{Z}[a, a^{-1}]$-linear map

$$\begin{aligned} f : \mathcal{S}^{br} &\longrightarrow \mathbb{Z}[a, a^{-1}] \\ T &\longmapsto br(T) \begin{bmatrix} \langle L(0) \rangle \\ \langle L(\infty) \rangle \end{bmatrix} \end{aligned} \tag{4.1}$$

where

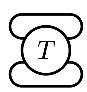
$L(0) = L\Big($  $\Big)$ and $L(\infty) = L\Big($  $\Big)$.

This map is simply an evaluation map computing the bracket of $L(T)$ since

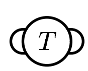
$$\langle L(T) \rangle = br(T) \begin{bmatrix} \langle L(0) \rangle \\ \langle L(\infty) \rangle \end{bmatrix} = f(T).$$

Given a tangle T, one may form a link in a number of ways by choosing an external wiring. As with the previous construction, there are only two such external wirings which do not produce any new crossings.

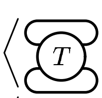
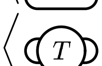
Definition 4.4. *For any Conway tangle* T *we may form the* numerator closure

$T^N =$ 

and the denominator closure

$T^D =$  .

Now returning to the link $L(T)$, recall that the external wiring $L \setminus T$ is itself a tangle. Again, all crossings and closed loops may be eliminated using the bracket axioms so that

$\langle L \setminus T \rangle = br(L \setminus T) \Bigg[\Big\langle$  $\Big\rangle$ / $\Big\langle$  $\Big\rangle\Bigg]$

$$= \begin{bmatrix} \langle T^N \rangle & \langle T^D \rangle \end{bmatrix} br(L \setminus T)^{\top}$$

This gives rise to another $\mathbb{Z}[a, a^{-1}]$-linear evaluation map

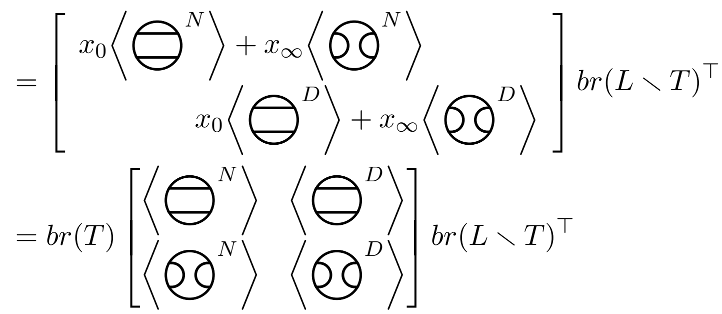
$$\begin{aligned} \mathcal{S}^{br} &\longrightarrow \mathbb{Z}[a, a^{-1}] \\ L \setminus T &\longmapsto \begin{bmatrix} \langle T^N \rangle & \langle T^D \rangle \end{bmatrix} br(L \setminus T)^\top \end{aligned} \tag{4.2}$$

In fact, combining the linear maps (4.1) and (4.2) forms a bilinear map

$$\begin{aligned} F : \mathcal{S}^{br} \times \mathcal{S}^{br} &\longrightarrow \mathbb{Z}[a, a^{-1}] \\ (T, L \setminus T) &\longmapsto \langle L(T) \rangle \end{aligned} \tag{4.3}$$

where

$$\langle L(T) \rangle = \begin{bmatrix} \langle T^N \rangle & \langle T^D \rangle \end{bmatrix} br(L \setminus T)^\top$$



$$= br(T) \begin{bmatrix} \delta & 1 \\ 1 & \delta \end{bmatrix} br(L \setminus T)^\top$$

so that given $T, U \in \mathcal{S}^{br}$ we have

$$F(T, U) = br(T) \begin{bmatrix} \delta & 1 \\ 1 & \delta \end{bmatrix} br(U)^\top.$$

Definition 4.5. *For Conway tangles T and U define the link $J(T, U) = (TU)^N$ and call this the* join *of T and U.*

We have that

$$\langle J(T, U) \rangle = F(T, U)$$

by definition, and further

$$L(T) \sim J(T, L \setminus T).$$

Definition 4.6. *For Conway tangles T and U define the* connected sum *$T^D \# U^D = (TU)^D$.*

Since any link L may be written as T^D for some tangle T this definition gives rise to a connected sum for links. It follows that

$$\langle L_1 \# L_2 \rangle = \langle L_1 \rangle \langle L_2 \rangle,$$

and

$$V_{L_1 \# L_2} = V_{L_1} V_{L_2}$$

provided orientations agree. A similar argument gives such an equality for the HOMFLY polynomial, and hence the Alexander polynomial as well.

4.2 Conway Mutation

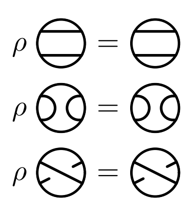
Consider a link diagram containing some Conway tangle T. We can choose the coordinate system so that T is contained in the unit disk, for convenience of notation. Further, we can arrange that the 4 points of intersection between the link and the boundary of the disk are

$$\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0 \right).$$

Let ρ be a 180 degree rotation of the unit disk about any of the three coordinate axis. Note that ρ leaves the external wiring unchanged and, for such a projection, ρ fixes the boundary points as a set.

Definition 4.7. *Given a link $L(T)$ where $T \in \mathcal{S}^{br}$ define the* Conway mutant *denoted by $L(\rho T)$.*

Notice that



so that

$$\begin{aligned}\langle L(T) \rangle &= br(T) \begin{bmatrix} \langle L(0) \rangle \\ \langle L(\infty) \rangle \end{bmatrix} \\ &= br(\rho T) \begin{bmatrix} \langle L(0) \rangle \\ \langle L(\infty) \rangle \end{bmatrix} \\ &= \langle L(\rho T) \rangle.\end{aligned}$$

Moreover, with orientation dictated by the external wiring

$$w(T) = w(\rho T)$$

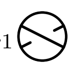
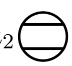
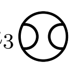
so we have the following theorem.

Theorem 4.8. $V_{L(T)} = V_{L(\rho T)}$.

While it may be that $L(T) \not\approx L(\rho T)$, it is certain that this method does not provide an answer to question 3.2: It can be shown that a Conway mutant of the unknot is always unknotted [30]. Theorem 4.8 is in fact a corollary of a stronger statement.

Theorem 4.9. $P_{L(T)} = P_{L(\rho T)}$.

PROOF. Using the skein relation (3.6) defining the HOMFLY polynomial, it is possible to decompose any tangle T into a linear combination of the form

$T = a_1$  $+ a_2$  $+ a_3$ 

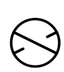
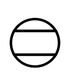
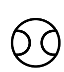
where $a_i \in \mathbb{Z}[t^{\pm 1}, x^{\pm 1}]$. Therefore, these tangles provide a basis for presenting the HOMFLY polynomial of a tangle T. Thus, we can define a $\mathbb{Z}[t^{\pm 1}, x^{\pm 1}]$-module $\mathcal{S}^P$ generated by isotopy classes of tangles up to equivalence under the skein relation. Moreover, if $L = L(T)$ then we have a $\mathbb{Z}[t^{\pm 1}, x^{\pm 1}]$-linear evaluation map

$$\begin{aligned} \mathcal{S}^P &\longrightarrow \mathbb{Z}[t^{\pm 1}, x^{\pm 1}] \\ T &\longmapsto P_{L(T)} \end{aligned}$$

or, more generally, the bi-linear evaluation map

$$\begin{aligned} \mathcal{S}^P \times \mathcal{S}^P &\longrightarrow \mathbb{Z}[t^{\pm 1}, x^{\pm 1}] \\ (T, U) &\longmapsto P_{J(T,U)}. \end{aligned}$$

Since the basis

$\{$  ,  ,  $\}$

is ρ-invariant, it follows that P_T and $P_{\rho T}$ are equal hence

$$\begin{array}{ccc} \mathcal{S}^P \times \mathcal{S}^P & \searrow & \\ \downarrow{\scriptstyle \rho \times id} & & \mathbb{Z}[t^{\pm 1}, x^{\pm 1}] \\ \mathcal{S}^P \times \mathcal{S}^P & \nearrow & \end{array}$$

commutes and

$$P_{L(T)} = P_{L(\rho T)}.$$

$\square$

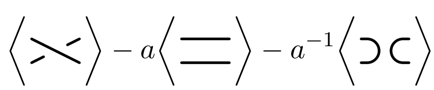
4.3 The Skein Module

Everything that has been said regarding tangles to this point can be stated in a more general setting [30, 31].

Definition 4.10. *Given a link L in $\mathbb{S}^3$ consider a 3-ball $B^3 \subset \mathbb{S}^3$ such that ∂B^3 intersects L in exactly $2n$ points. The intersection $B^3 \cap L$ is called an* n-tangle *denoted by T. As before, the exterior of the n-tangle $\overline{\mathbb{S}^3 \setminus B^3 \cap L}$ is another n-tangle $L \setminus T$ called an* external wiring.

In this setting, Conway tangles arise for $n = 2$ as 2-tangles.

Let $\mathcal{M}_n$ be the (infinitely generated) free $\mathbb{Z}[a, a^{-1}]$-module generated by the set of equivalence classes of n-tangles. The axioms (3.2) and (3.3) defining the bracket give rise to an ideal $\mathcal{I}_n \subset \mathcal{M}_n$ generated by

 (4.4)

$$\left\langle T \sqcup \bigcirc \right\rangle - \delta \left\langle T \right\rangle \tag{4.5}$$

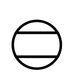
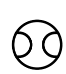
where $\delta = -a^{-2} - a^2$ and the $\langle\ \rangle$ indicate that the rest of the tangle is left unchanged.

Definition 4.11. *The $\mathbb{Z}[a, a^{-1}]$-module*

$$\mathcal{S}_n = {\mathcal{M}_n}\Big/{\mathcal{I}_n}$$

is called the (Kauffman bracket) skein module*. Note that $\mathcal{S}_2 = \mathcal{S}^{br}$.*

Due to the form of $\mathcal{I}_n$ it is possible to choose representatives for each equivalence class in $\mathcal{S}_n$ that have neither crossings nor closed loops. These tangles form a basis for $\mathcal{S}_n$. We have seen, for example, that $\mathcal{S}_2$ is 2-dimensional as a module, with basis given by the fundamental Conway tangles

 and .

Theorem 4.12. *$\mathcal{S}_n$ has dimension*

$$C_n = \frac{(2n)!}{n!(n+1)!}$$

as a module.

PROOF. Simply put, we need to determine how many n-tangles there are that have no crossings or closed loops. That is, given a disk in the plane with $2n$ marked points on the boundary, how many ways can the points be connected by non-intersecting arcs (up to isotopy)?

Clearly, $C_1 = 1$, and as discussed earlier $C_2 = 2$.

Now suppose $n > 2$ and consider a disk with $2n$ points on the boundary. Starting at some chosen boundary point and numbering clockwise, the point labeled 1 must connect to an even labeled point, say $2k$. This arc divides the disk in two: One disk having $2(k-1)$ marked points, the other with $2(n-k)$. Therefore

$$\begin{aligned} C_n &= \sum_{k=1}^{n} C_{k-1}C_{n-k} \\ &= C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 \end{aligned}$$

where $C_0 = 1$ by convention. Now consider the generating function

$$f(x) = \sum_{i=0}^{\infty} C_i x^i$$

and notice

$$(f(x))^2 = \sum_{i=0}^{\infty} \left(\sum_{k=1}^{i} C_{k-1}C_{n-k} \right) x^i$$

so that

$$x(f(x))^2 = f(x) - 1$$

and

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

To deduce the coefficients of $f(x)$, first consider the expansion of $\sqrt{z}$ about 1.

$$\sqrt{z} = \sum_{i=0}^{\infty} d_i (z-1)^i$$

where

$$d_i = \begin{cases} 1 & i = 0 \\ \frac{(-1)^{i-1}(2i-3)!!}{2^i i!} & i > 0 \end{cases}.$$

Therefore with $z = 1 - 4x$

$$\begin{aligned}\sqrt{1-4x} &= \sum_{i=0}^{\infty} d_i(-4x)^i \\ &= 1 + \sum_{i=1}^{\infty}(-1)^i 2^i 2^i d_i x^i\end{aligned}$$

and

$$\begin{aligned}f(x) &= -\frac{1}{2x}\sum_{i=1}^{\infty}(-1)^i 2^i 2^i d_i x^i \\ &= \sum_{i=1}^{\infty}(-1)^{i-1}2^i 2^{i-1} d_i x^{i-1}\end{aligned}$$

so that

$$\begin{aligned}C_i &= (-1)^i 2^{i+1} 2^i d_{i+1} \\ &= (-1)^i - 2^{i+1} 2^i \frac{(-1)^i(2i-1)!!}{2^{i+1}(i+1)!} \\ &= \frac{2^i(2i-1)!!}{(i+1)!}\end{aligned}$$

for $i > 1$. Finally, the fact that

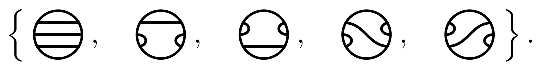
$$\frac{2^n(2n-1)!!}{(n+1)!} = \frac{(2n)!}{n!(n+1)!}$$

follows by induction since

$$\begin{aligned}\frac{2^{n+1}(2(n+1)-1)!!}{(n+2)!} &= \frac{2(2n+1)}{n+2}\frac{2^n(2n-1)!!}{(n+1)!} \\ &= \frac{2(n+1)(2n+1)}{(n+2)(n+1)}C_n \\ &= \frac{(2n+2)(2n+1)}{(n+2)(n+1)}\frac{(2n)!}{n!(n+1)!} \\ &= \frac{(2(n+1))!}{(n+1)!(n+2)!}.\end{aligned}$$

□

As an example of theorem 4.12, the 5-dimensional module $\mathcal{S}_3$ has basis given by



(4.6)

Let a given n-tangle diagram be contained in the unit disk so that, of the $2n$ boundary points, n have positive x-coordinate while the remaining n have negative x-coordinate. With this special position, multiplication of tangles by concatenation, as introduced for Conway tangles, extends to all n-tangles. When two n-tangles are in fact n-braids, we are reduced to multiplication in B_n. With this multiplication, $\mathcal{S}_n$ has an algebra structure called the *Temperly-Lieb algebra* [21, 22].

The n-dimensional Temperly-Lieb algebra $\mathcal{TL}_n$ over $\mathbb{Z}[a, a^{-1}]$ has generators $e_1, e_2, \ldots, e_{n-1}$ and relations

$$e_i^2 = \delta e_i \tag{4.7}$$

$$e_i e_j = e_j e_i \qquad \text{for} \quad |i-j| > 1 \tag{4.8}$$

$$e_i e_j e_i = e_i \qquad \text{for} \quad |i-j| = 1. \tag{4.9}$$

The multiplicative identity for this algebra is exactly the identity in B_n, and the generators are tangles of the form shown in figure 4.2.

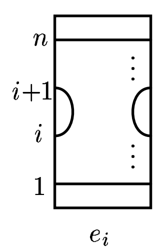


Figure 4.2: The generator e_i

For example, $\mathcal{TL}_3$ is generated by $\{e_1, e_2\}$, while the basis for the module $\mathcal{S}_3$ is the set of elements $\{1, e_1, e_2, e_1e_2, e_2e_1\}$ as in (4.6).

Notice that there is a representation of the braid group via the Kauffman bracket given by

$$\begin{aligned} B_n &\longrightarrow \mathcal{TL}_n \\ \sigma_i &\longmapsto a + a^{-1}e_i \\ \sigma_i^{-1} &\longmapsto ae_i + a^{-1}. \end{aligned}$$

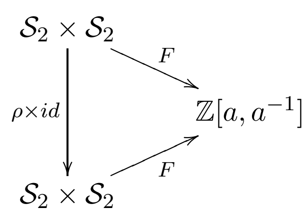
4.4 Linear Maps

Let $L = L(T_1, \ldots, T_k)$ be a link where $\{T_i\}$ is a collection of subtangles $T_i \subset L$. If T_i' is a tangle such that $T_i = T_i'$ as elements of $\mathcal{S}_n$ then, in the most general setting, $L' = L(T_1', \ldots, T_k')$ is a *mutant* of L (relative to the Kauffman bracket). Therefore when $w(L) = w(L')$, we have that

$$V_L = V_{L'}.$$

Of course, it may be that $L \not\approx L'$ and this approach has been used in attempts to answer question 3.2 [30, 31].

Let's first revisit Conway mutation in this context. We have, given the bilinear evaluation map F and a 180 degree rotation ρ, the commutative diagram



since $F = F \circ (\rho \times id)$. We saw that the linear transformation ρ was in fact the identity transformation on $\mathcal{S}_2$, and as a result the link $L = J(T, U)$ and the mutant $L' = J(\rho T, U)$ have the same Kauffman bracket.

A possible generalization arises naturally at this stage. As was pointed out earlier, it is possible to construct a link from two tangles in many different and complicated ways. Starting with $T \subset B_T^3 \subset \mathbb{S}^3$ and $U \subset B_U^3 \subset \mathbb{S}^3$, the link $L(T, U)$ is constructed by choosing an external wiring of $\mathbb{S}^3 \smallsetminus (B_T^3 \cup B_U^3)$. In this setting we have

$$\begin{aligned}\langle L(T,U)\rangle &= br(T)\begin{bmatrix}\langle L(0,U)\rangle\\ \langle L(\infty,U)\rangle\end{bmatrix}\\ &= br(T)\begin{bmatrix}\langle L(0,0)\rangle & \langle L(0,\infty)\rangle\\ \langle L(\infty,0)\rangle & \langle L(\infty,\infty)\rangle\end{bmatrix} br(U)^\top\\ &= br(T)\,\mathcal{L}\,br(U)^\top\end{aligned}$$

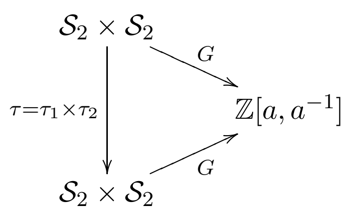
which gives rise to the bilinear map

$$\begin{aligned}G:\mathcal{S}_2\times\mathcal{S}_2 &\longrightarrow \mathbb{Z}[a,a^{-1}]\\ (T,U) &\longmapsto br(T)\,\mathcal{L}\,br(U)^\top.\end{aligned}$$

If additionally there is a linear transformation

$$\begin{aligned}\tau:\mathcal{S}_2\times\mathcal{S}_2 &\longrightarrow \mathcal{S}_2\times\mathcal{S}_2\\ (T,U) &\longmapsto (\tau_1 T,\tau_2 U)\end{aligned}$$

which acts as the identity as a linear transformation of modules, then we have the commutative diagram



and finally, if $w(L(T,U)) = w(L(\tau_1 T,\tau_2 U))$ we can conclude that

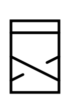
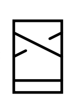
$$V_{L(T,U)} = V_{L(\tau_1 T,\tau_2 U)}.$$

4.5 Braid Actions

The three strand braid group has presentation

$$B_3 = \langle \sigma_1,\sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$$

where

$\sigma_1 =$  and $\sigma_2 =$ .

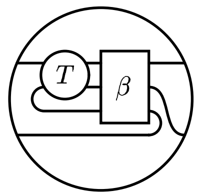


Figure 4.3: The tangle $T^\beta \in \mathcal{S}_2$

Given $T \in \mathcal{S}_2$ and a $\beta \in B_3$ we can define a new 2-tangle denoted T^β as in figure 4.3.

Proposition 4.13. *The map*

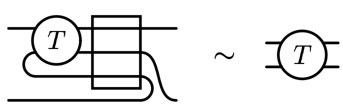
$$\begin{aligned} \mathcal{S}_2 \times B_3 &\longrightarrow \mathcal{S}_2 \\ (T, \beta) &\longmapsto T^\beta \end{aligned}$$

is a well defined group action.

PROOF. Let id_{B_3} be the identity braid. Then for any tangle T we have

$$T^{id_{B_3}} = T$$

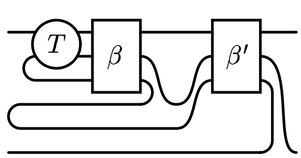
since



For $\beta, \beta' \in B_3$, the product $\beta\beta'$ is defined by concatenation so that

$$(T^\beta)^{\beta'} = T^{\beta\beta'}$$

by planar isotopy of the diagram



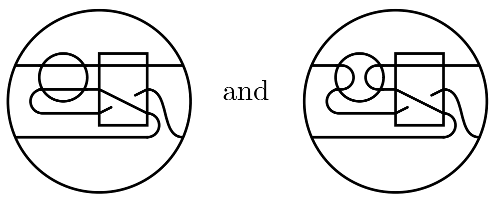
□

Proposition 4.14. *For $T \in \mathcal{S}_2$ and $\beta \in B_3$*

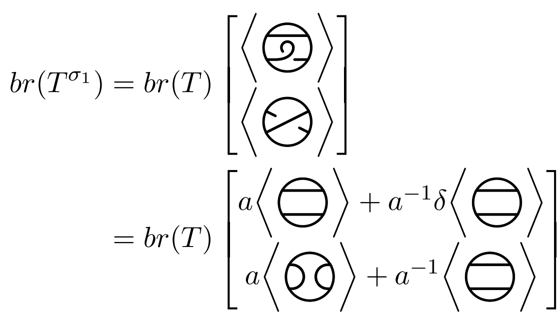
$$br(T^{\sigma_1}) = br(T)\begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix} \tag{4.10}$$

$$br(T^{\sigma_2}) = br(T)\begin{bmatrix} a & a^{-1} \\ 0 & -a^{-3} \end{bmatrix}. \tag{4.11}$$

PROOF. Applying the action of σ_1 to an arbitrary tangle T, we may decompose T^{σ_1} into the tangles



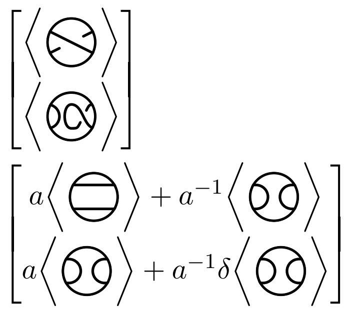
in $\mathcal{S}_2$. Relaxing these diagrams gives rise to the following computation in $\mathcal{S}_2$:



$$= br(T)\begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix}$$

Similarly, for the action of σ_2:

$$br(T^{\sigma_2}) = br(T)$$



$$= br(T) \begin{bmatrix} a & a^{-1} \\ 0 & -a^{-3} \end{bmatrix}$$

□

This gives rise to a group homomorphism

$$\begin{aligned} \Phi : B_3 &\longrightarrow GL_2(\mathbb{Z}[a, a^{-1}]) \\ \sigma_1 &\longmapsto \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix} \\ \sigma_2 &\longmapsto \begin{bmatrix} a & a^{-1} \\ 0 & -a^{-3} \end{bmatrix} \end{aligned}$$

since

$$\begin{aligned} \Phi(\sigma_1\sigma_2\sigma_1) &= \begin{bmatrix} -a^{-2} & -a^{-4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix} \\ &= \begin{bmatrix} 0 & -a^{-3} \\ -a^{-3} & 0 \end{bmatrix} \\ &= \begin{bmatrix} a & a^{-1} \\ 0 & -a^{-3} \end{bmatrix} \begin{bmatrix} -a^{-2} & -a^{-4} \\ 1 & 0 \end{bmatrix} \\ &= \Phi(\sigma_2\sigma_1\sigma_2). \end{aligned}$$

Question 4.15. *Is this representation of B_3 faithful?*

With the B_3-action on $\mathcal{S}_2$, consider the linear transformation given by

$$\begin{aligned} \beta : \mathcal{S}_2 \times \mathcal{S}_2 &\longrightarrow \mathcal{S}_2 \times \mathcal{S}_2 \\ (T, U) &\longmapsto (T^{\beta}, U^{\beta^{-1}}). \end{aligned}$$

For a link of the form $L(T, U)$, this leads to the definition of a new link $L(T^\beta, U^{\beta^{-1}})$.

Denote the *evaluation matrix* of $L(T, U)$ by

$$\mathcal{L} = \begin{bmatrix} \langle L(0,0)\rangle & \langle L(0,\infty)\rangle \\ \langle L(\infty,0)\rangle & \langle L(\infty,\infty)\rangle \end{bmatrix}$$

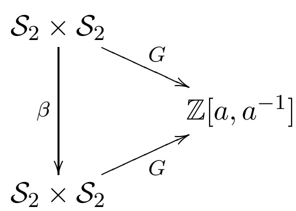
and suppose that $\mathcal{L} \in GL_2(\mathbb{Z}[a, a^{-1}])$, that is, $\det(\mathcal{L}) \neq 0$. Note that

$$\begin{aligned} \langle L(T,U)\rangle &= br(T)\,\mathcal{L}\,br(U)^\top \\ \langle L(T^\beta, U^{\beta^{-1}})\rangle &= br(T)\Phi(\beta)\,\mathcal{L}\,(\Phi(\beta^{-1}))^\top br(U)^\top. \end{aligned}$$

So, defining a second B_3-action

$$\begin{aligned} B_3 \times GL_2(\mathbb{Z}[a, a^{-1}]) &\longrightarrow GL_2(\mathbb{Z}[a, a^{-1}]) \\ (\beta, \mathcal{L}) &\longmapsto \Phi(\beta)\,\mathcal{L}\,(\Phi(\beta^{-1}))^\top, \end{aligned}$$

we are led to an algebraic question. When a non-trivial $\beta \in B_3$ gives rise to a fixed point under this action, the linear transformation given by β is the identity. Thus $G = G \circ \beta$ and we have the commutative diagram



where $\mathcal{L} \in \text{Fix}(\beta)$, so that

$$\langle L(T,U)\rangle = \langle L(T^\beta, U^{\beta^{-1}})\rangle.$$

In particular, we would like to study the case where

$$L(T,U) \nsim L(T^\beta, U^{\beta^{-1}}).$$

Question 4.16. *For a given link $L(T, U)$ with evaluation matrix $\mathcal{L} \in GL_2(\mathbb{Z}[a, a^{-1}])$, what are the elements $\beta \in B_3$ such that $\mathcal{L} \in \text{Fix}(\beta)$?*

This question is the main focus of the following chapters.

Chapter 5

Kanenobu Knots

5.1 Construction

Shortly after the discovery of the HOMFLY polynomial, Kanenobu introduced families of distinct knots having the same HOMFLY polynomial and hence the same Jones and Alexander polynomials as well [18]. It turns out that these knots are members of a much larger class of knots which we will denote by $K(T,U)$ for tangles $T, U \in \mathcal{S}_2$.

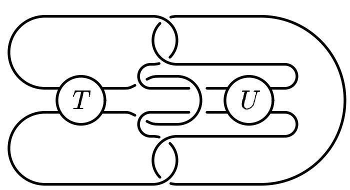


Figure 5.1: The Kanenobu knot $K(T,U)$

Proposition 5.1. *Suppose x is a non-trivial polynomial in $\mathbb{Z}[a, a^{-1}]$ so that*

$$\mathcal{X} = \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix} \in GL(\mathbb{Z}[a, a^{-1}])$$

where $\delta = -a^{-2} - a^2$. Then $\Phi(\sigma_2)\,\mathcal{X}\,\Phi(\sigma_2^{-1})^\top = \mathcal{X}$ and $\mathcal{X} \in \mathrm{Fix}(\sigma_2)$ under the B_3-action on $GL(\mathbb{Z}[a, a^{-1}])$.

PROOF. Since

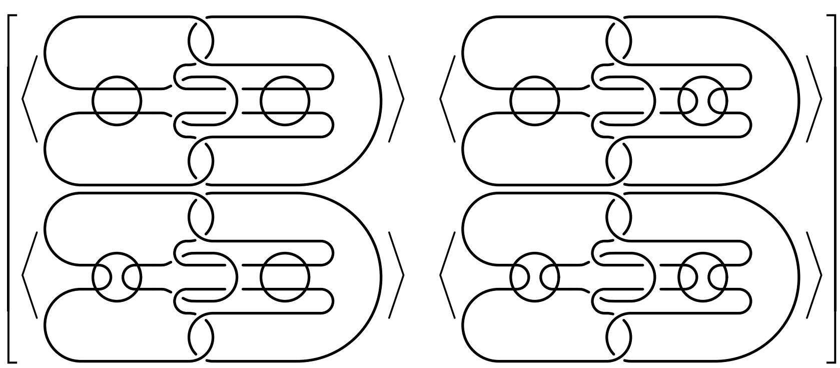
$$\Phi(\sigma_2) = \begin{bmatrix} a & a^{-1} \\ 0 & -a^{-3} \end{bmatrix} \quad \text{and} \quad \Phi(\sigma_2^{-1}) = \begin{bmatrix} a^{-1} & a \\ 0 & -a^{3} \end{bmatrix}$$

we have

$$\begin{aligned}
\begin{bmatrix} a & a^{-1} \\ 0 & -a^{-3} \end{bmatrix}\begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix}\begin{bmatrix} a^{-1} & 0 \\ a & -a^{3} \end{bmatrix} &= \begin{bmatrix} ax + a^{-1}\delta & a\delta + a^{-1}\delta^2 \\ -a^{-3}\delta & -a^{-3}\delta^2 \end{bmatrix}\begin{bmatrix} a^{-1} & 0 \\ a & -a^{3} \end{bmatrix} \\
&= \begin{bmatrix} x + a^{-2}\delta + a^2\delta + \delta^2 & -a^4\delta - a^2\delta^2 \\ -a^{-4}\delta - a^{-2}\delta^2 & \delta^2 \end{bmatrix} \\
&= \begin{bmatrix} x + \delta(a^{-2} + a^2 + \delta) & \delta(-a^4 - a^2\delta) \\ \delta(-a^{-4} - a^{-2}\delta) & \delta^2 \end{bmatrix} \\
&= \begin{bmatrix} x & \delta(-a^4 + 1 + a^4) \\ \delta(-a^{-4} + a^{-4} + 1) & \delta^2 \end{bmatrix} \\
&= \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix}
\end{aligned}$$

with $\delta = -a^{-2} - a^2$. □

To compute the bracket of the Kanenobu knot $K(T, U)$, we need the evaluation matrix

$\mathcal{K} =$  .

Since $K(0,0)$ is the connected sum of two figure eight knots (the figure eight is denoted 4_1 as in [32]), we can compute

$$\left\langle 4_1 \right\rangle = \left\langle \overline{\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}} \right\rangle$$

using the Temperly-Lieb algebra $\mathcal{TL}_3$. We have

$$\sigma_2\sigma_1^{-1} = 1 + a^2e_1 + a^{-2}e_2 + e_2e_1$$

as an element of $\mathcal{TL}_3$ so that

$$\begin{aligned}
\left(\sigma_2\sigma_1^{-1}\right)^2 &= (1 + a^2e_1 + a^{-2}e_2 + e_2e_1)^2 \\
&= 1 + (3a^2 + a^4\delta)e_1 + (3a^{-2} + a^{-4}\delta)e_2 + e_1e_2 + (4 + a^2\delta + a^{-2}\delta)e_2e_1.
\end{aligned}$$

Then the Kauffman bracket is given by

$$\begin{aligned}\left\langle \overline{\left(\sigma_2\sigma_1^{-1}\right)^2} \right\rangle &= \delta^2 + (3a^2 + a^4\delta)\delta + (3a^{-2} + a^{-4}\delta)\delta + 1 + (4 + a^2\delta + a^{-2}\delta)(1) \\ &= \delta(-a^{-6} + 2a^{-4} + 2a^4 - a^6) + 5 \\ &= a^{-8} - a^{-4} + 1 - a^4 + a^8\end{aligned}$$

and

$$\begin{aligned}\langle 4_1 \# 4_1 \rangle &= \left(a^{-8} - a^{-4} + 1 - a^4 + a^8\right)^2 \\ &= a^{-16} - 2a^{-12} + 3a^{-8} - 4a^{-4} + 5 - 4a^4 + 3a^8 - 2a^{12} + a^{16}.\end{aligned}$$

In addition, it can be seen from the braid closure $\overline{\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}} \sim 4_1$ that $w(4_1) = 0$ and hence

$$w\left(K(0,0)\right) = w(4_1 \# 4_1) = 0.$$

Thus the Jones polynomial of $K(0,0)$ is given by

$$V_{K(0,0)} = a^{-16} - 2a^{-12} + 3a^{-8} - 4a^{-4} + 5 - 4a^4 + 3a^8 - 2a^{12} + a^{16}.$$

Now the evaluation matrix for $K(T,U)$ is given by

$$\mathcal{K} = \begin{bmatrix} (a^{-8} - a^{-4} + 1 - a^4 + a^8)^2 & \delta \\ \delta & \delta^2 \end{bmatrix}$$

since the three entries for $K(0,\infty)$, $K(\infty,0)$ and $K(\infty,\infty)$ are all equivalent to unlinks with no crossings via applications of the Reidemeister move R_2 (recall that R_2 leaves the Kauffman bracket unchanged).

For any tangle diagram T, denote by $T^\star$ the tangle diagram obtained by switching each crossing of T. That is, for any choice of orientation

$$w(T^\star) = -w(T).$$

This can be extended to knot diagrams K, where $K^\star$ is the diagram such that

$$w(K^\star) = -w(K)$$

so that $K^\star$ is the mirror image of K.

When $U = T^\star$,

$$w(K(T,U)) = 0$$

the bilinear evaluation map for the bracket

$$G(T,U) = br(T)\,\mathcal{K}\,br(U)^\top$$

computes the Jones polynomial

$$V_{K(T,U)} = br(T)\,\mathcal{K}\,br(U)^\top.$$

Since $\mathcal{K}$ is of the form given in proposition 5.1, the bilinear map defined by the braid $\sigma_2^n \in B_3$

$$\begin{aligned}\sigma_2^n : \mathcal{S}_2 \times \mathcal{S}_2 &\to \mathcal{S}_2 \times \mathcal{S}_2 \\ (T,U) &\mapsto (T^{\sigma_2^n}, U^{\sigma_2^{-n}})\end{aligned}$$

is the identity transformation for every $n \in \mathbb{Z}$. Moreover, when $U = T^\star$

$$w\left(K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})\right) = 0$$

so we have the following theorem.

Theorem 5.2. *When $U = T^\star$, the family of knots given by*

$$K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})$$

for $n \in \mathbb{Z}$ are indistinguishable by the Jones polynomial.

Of course, that these are in fact distinct knots remains to be seen.

Kanenobu's original knot families [18] can be recovered from

$$K_{n,m} = K\left(T^{\sigma_2^{2n}}, T^{\sigma_2^{2m}}\right)$$

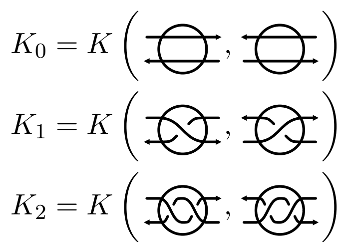
where $n, m \in \mathbb{Z}$ and T is the 0-tangle.

Theorem (Kanenobu). *$K_{n,m}$ and $K_{n',m'}$ have the same HOMFLY polynomial when $|n-m| = |n'-m'|$. Moreover, when (n,m) and (n',m') are pairwise distinct, these knots are distinct.*

The knots of Kanenobu's theorem are distinguished by their Alexander module structure [18].

5.2 Basic Examples

Consider the Kanenobu knots

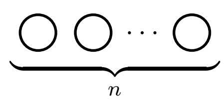


and notice that by applying the action of σ_2 we have

$$K_0 \xrightarrow{\sigma_2} K_1 \xrightarrow{\sigma_2} K_2$$

so that by construction, these knots have the same Jones polynomial.

First recall that the HOMFLY polynomial of the n-component unlink



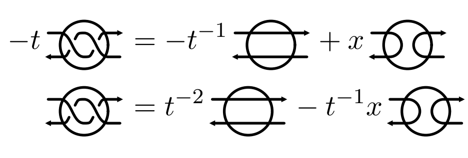
is given by

$$\left(\frac{t^{-1}-t}{x}\right)^{n-1}.$$

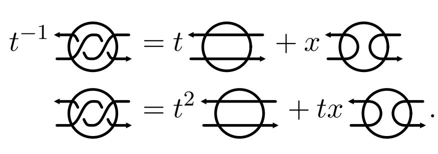
As this polynomial will be used often, we define

$$P_0 = \frac{t^{-1}-t}{x}.$$

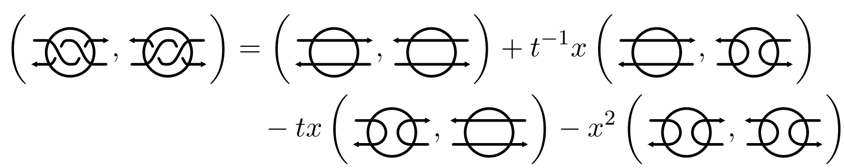
Now, using the skein relation (3.6) defining the HOMFLY polynomial $P(t, x)$, we can compute



and



Combining these tangles pairwise, we have



giving rise to equality among HOMFLY polynomials

$$\begin{aligned} P_{K_2} &= P_{K_0} + (t^{-1}x - tx)P_0 - x^2 P_0^2 \\ &= P_{K_0} + (t^{-1}x - tx)\frac{t^{-1} - t}{x} - x^2 \left(\frac{t^{-1} - t}{x}\right)^2 \\ &= P_{K_0}. \end{aligned}$$

This common polynomial[1] is

$$P_{K_0}(t, x) = (t^{-4} - 2t^{-2} + 3 - 2t^2 + t^4) + (-2t^{-2} + 2 - 2t^2)x^2 + x^4.$$

Notice that this is in agreement with Kanenobu's theorem.

On the other hand K_1 has HOMFLY polynomial

$$P_{K_1}(t, x) = (2t^{-2} - 3 + 2t^2) + (3t^{-2} - 8 + 3t^2)x^2 + (t^{-2} - 5 + t^2)x^4 - x^6$$

and we can conclude that K_0 and K_1 are distinct knots despite having the same Jones polynomial:

$$a^{-16} - 2a^{-12} + 3a^{-8} - 4a^{-4} + 5 - 4a^4 + 3a^8 - 2a^{12} + a^{16}$$

Further, applying theorem 4.9, these knots cannot be Conway mutants as they have different HOMFLY polynomials.

5.3 Main Theorem

Theorem 5.3. *For each 2-tangle T there exists a pair of external wirings for T that produce distinct links that have the same Jones polynomial. Moreover, the links obtained are not Conway mutants.*

[1] computed using KNOTSCAPE

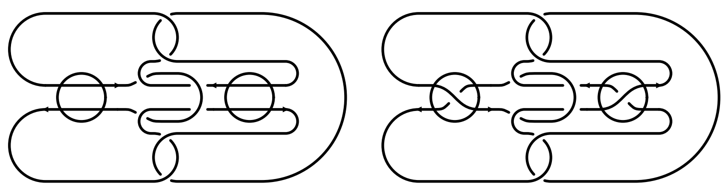


Figure 5.2: Distinct knots that are not Conway mutants

PROOF. Take $U = T^\star$ (that is, the tangle such that $w(U) = -w(T)$) and define Kanenobu knots for the pair (T, U)

$$K = K(T, U) \quad \text{and} \quad K^{\sigma_2} = K(T^{\sigma_2}, U^{\sigma_2}).$$

Then, by construction, we have that

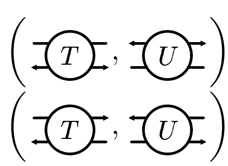
$$V_K = V_{K^{\sigma_2}}.$$

It remains to show that these are in fact distinct knots. To see this, we compute the HOMFLY polynomials P_K and $P_{K^{\sigma_2}}$.

Now with the requirement that the tangle $U = T^\star$, there are two choices of orientations for the tangles that are compatible with an orientation of the knot (or possibly link, in which case a choice of orientation is made) $K(T, U)$. They are

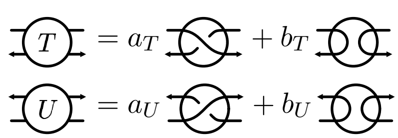
Type 1

Type 2

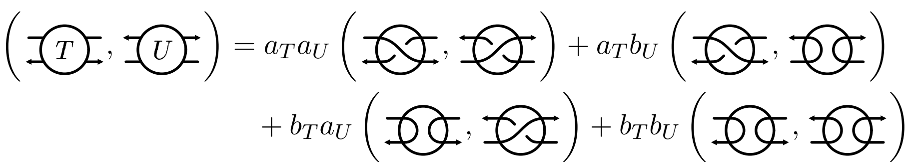


so we proceed in two cases.

Type 1. Using the skein relation we can decompose



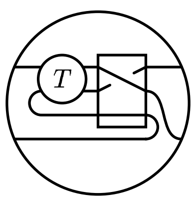
where $a_T, b_T, a_U, b_U \in \mathbb{Z}[t^{\pm}, x^{\pm}]$. Combining pairwise we obtain

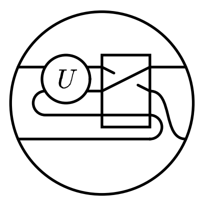


so that

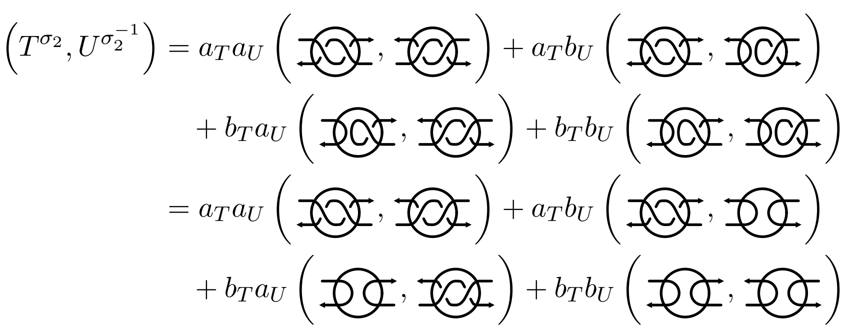
$$\begin{aligned} P_K &= a_Ta_UP_{K_1} + (a_Tb_U + b_Ta_U)P_0 + b_Tb_UP_0^2 \\ &= a_Ta_UP_{K_1} + (a_Tb_U + b_Ta_U)\left(\frac{t^{-1}-t}{x}\right) + b_Tb_U\left(\frac{t^{-1}-t}{x}\right)^2 \\ &= a_Ta_UP_{K_1} + R \end{aligned}$$

where $R = (a_Tb_U + b_Ta_U)\left(\frac{t^{-1}-t}{x}\right) + b_Tb_U\left(\frac{t^{-1}-t}{x}\right)^2$. Now applying the action of σ_2 we have

$T^{\sigma_2} =$ 

and $U^{\sigma_2^{-1}} =$ 

therefore



so that

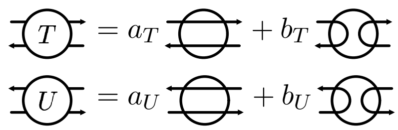
$$\begin{aligned} P_{K^{\sigma_2}} &= a_Ta_UP_{K_2} + (a_Tb_U + b_Ta_U)P_0 + b_Tb_UP_0^2 \\ &= P_{K_0} + R \end{aligned}$$

since $P_{K_2} = P_{K_0}$. However, as $P_{K_1} \neq P_{K_0}$ we have that

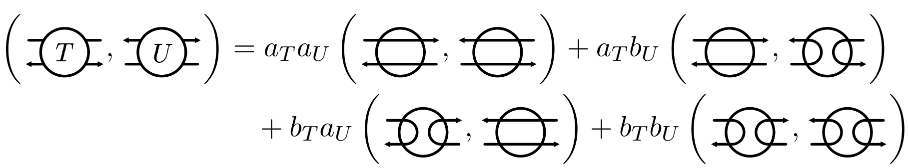
$$P_K \neq P_{K^{\sigma_2}}$$

giving rise to distinct knots.

Type 2. As before, the tangles are decomposed via the HOMFLY skein relation



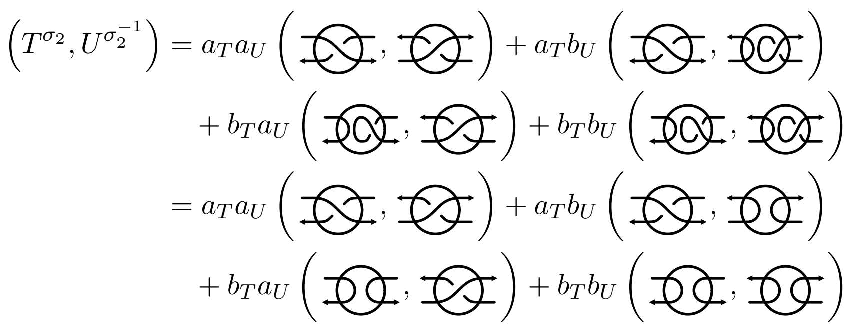
for some other $a_T, b_T, a_U, b_U \in \mathbb{Z}[t^{\pm}, x^{\pm}]$. Combining pairwise



and

$$\begin{aligned}
P_K &= a_T a_U P_{K_0} + (a_T b_U + b_T a_U) P_0 + b_T b_U P_0^2 \\
&= a_T a_U P_{K_0} + (a_T b_U + b_T a_U)\left(\frac{t^{-1} - t}{x}\right) + b_T b_U \left(\frac{t^{-1} - t}{x}\right)^2 \\
&= a_T a_U P_{K_0} + R
\end{aligned}$$

with $R \in \mathbb{Z}[t^{\pm}, x^{\pm}]$ as before. Again, applying the action of σ_2 we have



and the HOMFLY polynomial

$$\begin{aligned}
P_{K^{\sigma_2}} &= a_T a_U P_{K_1} + (a_T b_U + b_T a_U) P_0 + b_T b_U P_0^2 \\
&= a_T a_U P_{K_1} + R.
\end{aligned}$$

Once again, as $P_{K_1} \neq P_{K_0}$ we have that

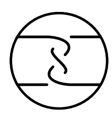
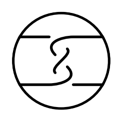
$$P_K \neq P_{K^{\sigma_2}}$$

giving rise to distinct knots.

Finally, as K and K^{σ_2} have distinct HOMFLY polynomials in both cases, it follows from theorem 4.9 that these knots cannot be Conway mutants. □

5.4 Examples

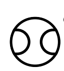
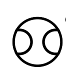
1. Consider the Kanenobu knot $K(T, U)$ where

$T =$  and $U =$ 

as in figure 5.3.

Definition 5.4. *A tangle is called* rational *if it is of the form T^β where $\beta \in B_3$ and the tangle T is either the 0-tangle or the ∞-tangle. This is equivalent to Conway's definition for rational tangles [8].*

As a result, the computation of the bracket for rational tangles is straightforward. In this example we have

$T =$  $^{\sigma_1^{-3}}$ and $U =$  $^{\sigma_1^{3}}$

so that

$$\begin{aligned} br(T) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix}^{-3} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -a^{3} & 0 \\ a & a^{-1} \end{bmatrix} \begin{bmatrix} -a^{3} & 0 \\ a & a^{-1} \end{bmatrix} \begin{bmatrix} -a^{3} & 0 \\ a & a^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -a^{9} & 0 \\ a^{-1} - a^{3} + a^{7} & a^{-3} \end{bmatrix} \\ &= \begin{bmatrix} a^{-1} - a^{3} + a^{7} & a^{-3} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} br(U) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix}^3 \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix} \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix} \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix} \\ &= \begin{bmatrix} a^{-7} - a^{-1} + a & a^3 \end{bmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \langle K(T,U) \rangle &= br(T)\,\mathcal{K}\,br(U)^\top \\ &= br(T) \begin{bmatrix} (a^{-8} - a^{-4} + 1 - a^4 + a^8)^2 & -a^{-2} - a^2 \\ -a^{-2} - a^2 & a^{-4} + 2 + a^4 \end{bmatrix} br(U)^\top \\ &= a^{-24} - 4a^{-20} + 10a^{-16} - 19a^{-12} + 27a^{-8} - 33a^{-4} \\ &\quad + 37 - 33a^4 + 27a^8 - 19a^{12} + 10a^{16} - 4a^{20} + a^{24} \end{aligned}$$

hence

$$\begin{aligned} V_{K(T,U)} =& a^{-24} - 4a^{-20} + 10a^{-16} - 19a^{-12} + 27a^{-8} - 33a^{-4} \\ & + 37 - 33a^4 + 27a^8 - 19a^{12} + 10a^{16} - 4a^{20} + a^{24}. \end{aligned}$$

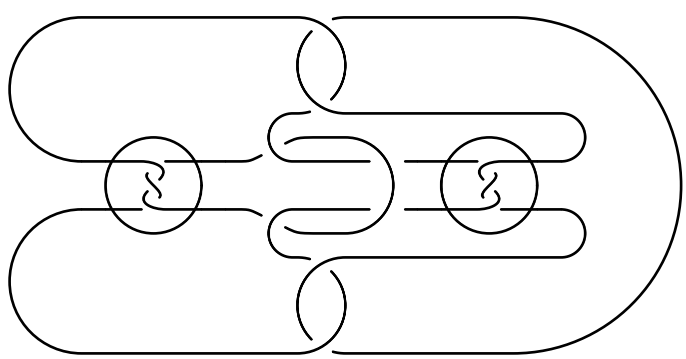


Figure 5.3: Example 1

This gives a collection of knots having the same Jones polynomial

$$V_{K(T,U)} = V_{K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})}$$

since $w(K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})) = 0$ for all $n \in \mathbb{Z}$.

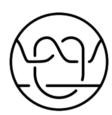
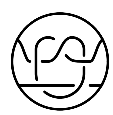
The Knots $K(T,U)$ and $K(T^{\sigma_2}, U^{\sigma_2^{-1}})$ are distinct as they have different HOMFLY polynomials[2]:

$$\begin{aligned} X_{K(T,U)} =& -1 + (6t^{-2} - 12 + 6t^2)x^2 + (9t^{-2} - 24 + 9t^2)x^4 \\ &+ (5t^{-2} - 19 + 5t^2)x^6 + (t^{-2} - 7 + t^2)x^8 - x^{10} \end{aligned}$$

$$\begin{aligned} X_{K(T^{\sigma_2}, U^{\sigma_2^{-1}})} =& (t^{-4} - 4t^{-2} + 7 - 4t^2 + t^4) \\ &+ (2t^{-4} - 7t^{-2} + 10 - 7t^2 + 2t^4)x^2 \\ &+ (t^{-4} - 6t^{-2} + 8 - 6t^2 + t^4)x^4 \\ &+ (-2t^{-2} + 4 - 2t^2)x^6 + x^8 \end{aligned}$$

In particular, these knots cannot be Conway mutants in view of theorem 4.9.

2. Now consider the case when T, U are not rational tangles. For this example take

$T =$  and $U =$ 

in $K(T,U)$. We can compute

$$br(T) = \begin{bmatrix} a^{-5} - 2a^{-1} + a^3 - a^7 & -a^{-11} + 2a^{-7} - 2a^{-3} + a \end{bmatrix}$$
$$br(U) = \begin{bmatrix} -a^{-7} + a^{-3} - 2a + a^5 & a^{-1} - 2a^3 + 2a^7 - a^{11} \end{bmatrix}$$

so that

$$\begin{aligned} \langle K(T,U) \rangle =& -a^{-28} + 5a^{-24} - 15a^{-20} + 31a^{-16} - 52a^{-12} + 73a^{-8} - 88a^{-4} \\ &+ 95 - 88a^4 + 73a^8 - 52a^{12} + 31a^{16} - 15a^{20} + 5a^{24} - a^{28}. \end{aligned}$$

Again we have a family of knots (distinct from those of example 1) such that

$$V_{K(T,U)} = V_{K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})}$$

since $w(K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})) = 0$ for all $n \in \mathbb{Z}$.

[2]computed using KNOTSCAPE

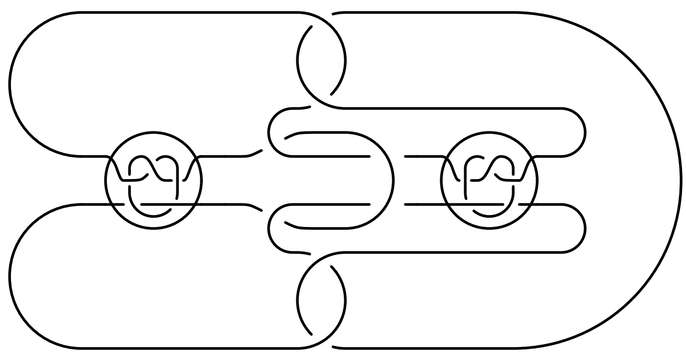


Figure 5.4: Example 2

The knots $K(T,U)$ and $K(T^{\sigma_2}, U^{\sigma_2^{-1}})$ of this example are distinct as can be seen by the HOMFLY polynomials[3]:

$$\begin{aligned}X_{K(T,U)} =&(-2t^{-6}+9t^{-4}-22t^{-2}+31-22t^2+9t^4-2t^6)\\&+(-3t^{-6}+22t^{-4}-58t^{-2}+82-58t^2+22t^4-3t^6)x^2\\&+(-t^{-6}+21t^{-4}-67t^{-2}+94-67t^2+21t^4-t^6)x^4\\&+(8t^{-4}-44t^{-2}+62-44t^2+8t^4)x^6\\&+(t^{-4}-15t^{-2}+28-15t^2+t^4)x^8\\&+(-2t^{-2}+8-2t^2)x^{10}+x^{12}\end{aligned}$$

$$\begin{aligned}X_{K(T^{\sigma_2},U^{\sigma_2^{-1}})} =&(-4t^{-4}+12t^{-2}-15+12t^2-4t^4)\\&+(-12t^{-4}+56t^{-2}-84+56t^2-12t^4)x^2\\&+(-13t^{-4}+99t^{-2}-176+99t^2-13t^4)x^4\\&+(-6t^{-4}+87t^{-2}-197+87t^2-6t^4)x^6\\&+(-t^{-4}+41t^{-2}-130+41t^2-t^4)x^8\\&+(10t^{-2}-51+10t^2)x^{10}+(t^{-2}-11+t^2)x^{12}-x^{14}\end{aligned}$$

Once again, from theorem 4.9 it follows that these knots are not related by Conway mutation.

[3]computed using KNOTSCAPE

5.5 Generalisation

We saw in proposition 5.1 under the action

$$B_3 \times GL_2(\mathbb{Z}[a, a^{-1}]) \longrightarrow GL_2(\mathbb{Z}[a, a^{-1}])$$

that

$$\left\{ \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix} \in GL_2\left(\mathbb{Z}[a, a^{-1}]\right) \right\} \subset \mathrm{Fix}(\sigma_2).$$

As a final task for this chapter, we'll define a family of links that generate such evaluation matricies.

Consider a slightly different diagram of the knot $K(T, U)$, given in figure 5.5.

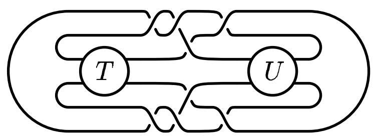


Figure 5.5: Another diagram of the Kanenobu knot $K(T, U)$

From this diagram, we are led to define a rather exotic braid closure that will be of use. That is, for the pair of tangles (T, U) and an appropriately chosen braid $\beta \in B_6$ we define a link $|\beta|$ as in figure 5.6.

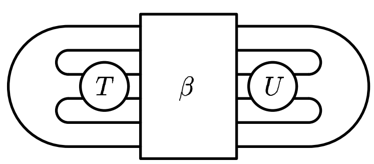


Figure 5.6: The link $|\beta|$

It remains to describe which braids in B_6 give rise to an evaluation matrix of the appropriate form. For this we will need two braid homomorphisms.

Let $N \gg n$ and define, for each non-negative $m \in \mathbb{Z}$, the inclusion homomorphism

$$\begin{aligned} i_m : B_n &\longrightarrow B_N \\ \sigma_k &\longmapsto \sigma_{k+m} \end{aligned}$$

for each $k \in \{1, \ldots, n-1\}$. Note that when $m = 0$, this is reduces to the natural inclusion $B_n < B_N$. Now the group $B_3 \oplus B_3$ arises as a subgroup of B_6 by choosing

$$\begin{aligned} B_3 \oplus B_3 &\longrightarrow B_6 \\ (\alpha, \beta) &\longmapsto i_0(\alpha) i_3(\beta). \end{aligned}$$

Notice that the image $i_0(\alpha)i_3(\beta)$ contains no occurrence of the generator σ_3 and hence

$$i_0(\alpha)i_3(\beta) = i_3(\beta)i_0(\alpha)$$

in B_6. Now define the *switch* homomorphism

$$\begin{aligned} s : B_3 &\longrightarrow B_3 \\ \sigma_1 &\longmapsto \sigma_2^{-1} \\ \sigma_2 &\longmapsto \sigma_1^{-1} \end{aligned}$$

and note that, given a 180 degree rotation ρ in the projection plane, $\rho(s\beta) = \beta^{-1}$.

Definition 5.5. *For each $\alpha \in B_3$ define the* Kanenobu braid $i_0(\alpha)i_3(s\alpha) \in B_6$.

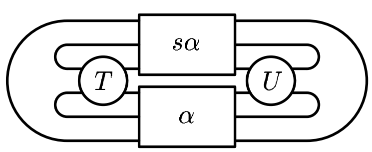


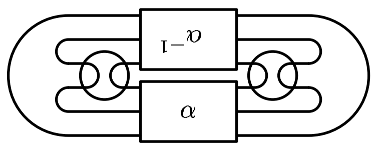
Figure 5.7: The closure of a Kanenobu braid

Theorem 5.6. *Let β be a Kanenobu braid. The evaluation matrix $\mathcal{X}$ associated with the link $|\beta|$ is an element of* $\mathrm{Fix}(\sigma_2)$.

PROOF. We need to compute

$$\mathcal{X} = \begin{bmatrix} \langle K(0,0) \rangle & \langle K(0,\infty) \rangle \\ \langle K(\infty,0) \rangle & \langle K(\infty,\infty) \rangle \end{bmatrix}$$

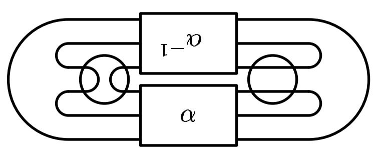
where $K(T,U) = |\beta|$ for some Kanenobu braid β. Since $\rho(s\alpha) = \alpha^{-1}$, the link $K(\infty,\infty)$ is of the form



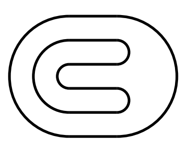
which is simply the (usual) braid closure $\overline{\alpha\alpha^{-1}}$. Hence

$$\left\langle K(\infty,\infty)\right\rangle = \left\langle \overline{\alpha\alpha^{-1}}\right\rangle = \delta^2$$

since the group operation $\sigma_i\sigma_i^{-1}$ coresponds to the Reidemeister move R_2, so that the Kauffman bracket is unchanged. Similarly, the link $K(\infty,0)$ (equivalently, $K(0,\infty)$) is of the form



which reduces, canceling $\alpha^{-1}\alpha$, to the link



so that

$$\langle K(\infty,0)\rangle = \langle K(0,\infty)\rangle = \delta.$$

Finally, let the polynomial $\langle K(0,0)\rangle = x$ so that

$$\mathcal{X} = \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix}$$

is an element of $\mathrm{Fix}(\sigma_2)$. □

5.6 More Examples

For the following examples, we introduce the shorthand $K(\sigma_2, \sigma_2^{-1})$ referring to the knot obtained from the action

$$K(0,0) \xrightarrow{\sigma_2} K(\sigma_2, \sigma_2^{-1}).$$

We continue numbering of examples from section 5.4.

3. Taking the braid $\sigma_1^3\sigma_2^{-1}\sigma_1 \in B_3$

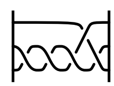


Figure 5.8: The braid $\sigma_1^3\sigma_2^{-1}\sigma_1$

we can form the Kanenobu braid $(\sigma_1^3\sigma_2^{-1}\sigma_1)(\sigma_5^{-3}\sigma_4\sigma_5^{-1}) \in B_6$

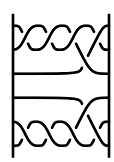


Figure 5.9: The Kanenobu braid $(\sigma_1^3\sigma_2^{-1}\sigma_1)(\sigma_5^{-3}\sigma_4\sigma_5^{-1})$

and the generalized Kanenobu knot

$$K(T,U) = \left|(\sigma_1^3\sigma_2^{-1}\sigma_1)(\sigma_5^{-3}\sigma_4\sigma_5^{-1})\right|.$$

$K(T,U)$ has evaluation matrix

$$\mathcal{K} = \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix}$$

where

$$\begin{aligned} x = &- a^{-20} + 2a^{-16} - 4a^{-12} + 6a^{-8} - 7a^{-4} \\ &+ 9 - 7a^4 + 6a^8 - 4a^{12} + 2a^{16} - a^{20}. \end{aligned}$$

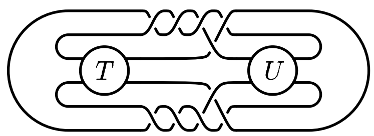


Figure 5.10: The knot $\left|(\sigma_1^3\sigma_2^{-1}\sigma_1)(\sigma_5^{-3}\sigma_4\sigma_5^{-1})\right|$

Since $w(K(T,U)) = 0$ when $U = T^\star$, the Jones Polynomial is given by

$$V_{K(T,U)} = br(T)\,\mathcal{K}\,br(U).$$

In particular,

$$V_{K(0,0)} = x$$

and we have that

$$V_{K(\sigma_2,\sigma_2^{-1})} = x.$$

It fact $K(0,0) \sim 5_2 \# 5_2^\star$ (following the notation in [32]).

These knots are distinct and not Conway mutants, as can be seen from the HOMFLY polynomial[4] of $K(0,0)$

$$(-4t^{-2} + 9 - 4t^2) + (-8t^{-2} + 20 - 8t^2)x^2 \\ + (-5t^{-2} + 18 - 5t^2)x^4 + (-t^{-2} + 7 - t^2)x^6 + x^8$$

while the HOMFLY polynomial of $K(\sigma_2, \sigma_2^{-1})$ is

$$(-t^{-4} + 3 - t^4) \\ + (-t^{-4} + t^{-2} + 4 - t^2 + t^4 - t^6)x^2 \\ + (t^{-2} + 2 + t^2)x^4$$

[4]computed using KNOTSCAPE

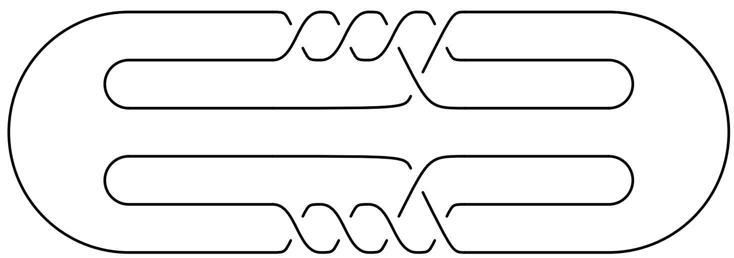


Figure 5.11: The knot $K(0,0) \sim 5_2 \# 5_2^\star$

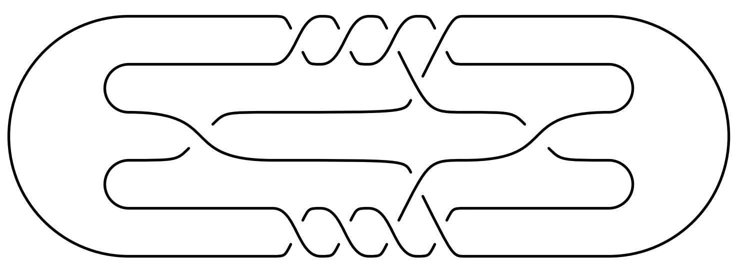


Figure 5.12: The knot $K(\sigma_2, \sigma_2^{-1})$

4. Taking the braid $\sigma_1^2\sigma_2^{-3}\sigma_1 \in B_3$

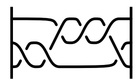


Figure 5.13: The braid $\sigma_1^2\sigma_2^{-3}\sigma_1$

we can form the Kanenobu braid $(\sigma_1^2\sigma_2^{-3}\sigma_1)(\sigma_5^{-2}\sigma_4^3\sigma_5^{-1}) \in B_6$

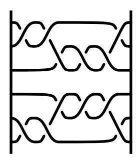


Figure 5.14: The Kanenobu braid $(\sigma_1^2\sigma_2^{-3}\sigma_1)(\sigma_5^{-2}\sigma_4^3\sigma_5^{-1})$

and the generalized Kanenobu knot

$$K(T,U) = \left|(\sigma_1^2\sigma_2^{-3}\sigma_1)(\sigma_5^{-2}\sigma_4^3\sigma_5^{-1})\right|.$$

$K(T,U)$ has evaluation matrix

$$\mathcal{K} = \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix}$$

where

$$\begin{aligned} x =& a^{-24} - 2a^{-20} + 4a^{-16} - 7a^{-12} + 9a^{-8} - 11a^{-4} \\ &+ 13 - 11a^4 + 9a^8 - 7a^{12} + 4a^{16} - 2a^{20} + a^{24}. \end{aligned}$$

Note that $K(0,0) \sim 6_1 \# 6_1^\star$ (following the notation in [32]).

Since $w(K(T,U)) = 0$ when $U = T^\star$, the Jones Polynomial is given by

$$V_{K(T,U)} = br(T)\,\mathcal{K}\,br(U).$$

In particular,

$$V_{K(0,0)} = x$$

and we have that

$$V_{K(\sigma_2,\sigma_2^{-1})} = x.$$

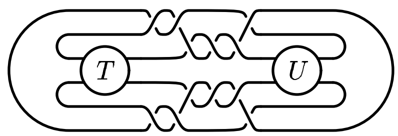


Figure 5.15: The knot $\left|(\sigma_1^2\sigma_2^{-3}\sigma_1)(\sigma_5^{-2}\sigma_4^3\sigma_5^{-1})\right|$

Once again we obtain distinct knots. The HOMFLY polynomial[5] of $K(0,0)$ is

$$\begin{aligned}&(4t^{-2}-7+4t^2)+(16t^{-2}-36+16t^2)x^2\\&\quad+(17t^{-2}-50+17t^2)x^4+(7t^{-2}-31+7t^2)x^6\\&\quad+(t^{-2}-1+t^2)x^8-x^{10}\end{aligned}$$

While the HOMFLY polynomial of $K(\sigma_2,\sigma_2^{-1})$ is

$$\begin{aligned}&(t^{-6}-t^{-4}-t^{-2}+3-t^2-t^4+t^6)\\&\quad+(-2t^{-4}-t^{-2}+2-t^2-2t^4)x^2+(t^{-2}+2+t^2)x^4\end{aligned}$$

Once again we have distinct knots that have the same Jones polynomial but are not related by Conway mutation.

[5]computed using KNOTSCAPE

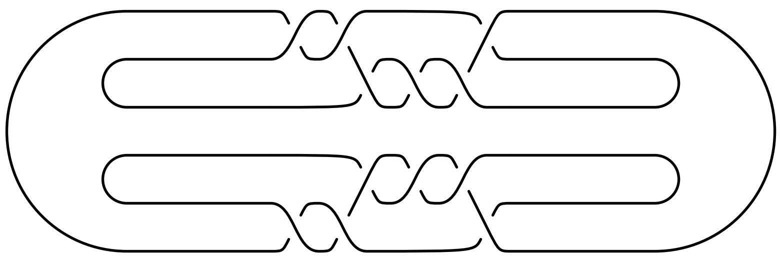


Figure 5.16: The knot $K(0,0) \sim 6_1 \# 6_1^{\star}$

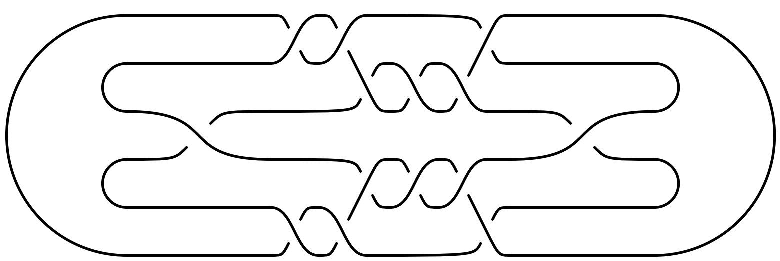


Figure 5.17: The knot $K(\sigma_2, \sigma_2^{-1})$

5.7 Observations

Definition 5.7. *If a link is equivalent to a 3-braid, closed as in figure 5.18, it is called a* 2-bridge *link.*

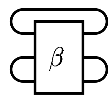


Figure 5.18: The 2-bridge link obtained from $\beta \in B_3$.

For a generalised Kanenobu knot $K(T,U)$, the knot $K(0,0)$ is always of the form $L \# L^\star$ where L is a 2-bridge link.

In the case where $K(0,0) \sim K \# K^\star$ is a connected sum of 2-bridge knots with more that 3 crossings, such a K is generated by taking the 2-bridge closure of an element $\alpha \in B_3$. Such a braid generates the Kanenobu braid $i_0(\alpha)i_2(s\alpha)$, and taking the closure

$$|i_0(\alpha)i_2(s\alpha)| = K(T,U)$$

with $U = T^\star$ gives rise to the evaluation matrix

$$\mathcal{K} = \begin{bmatrix} \langle K \# K^\star \rangle & \delta \\ \delta & \delta^2 \end{bmatrix}$$

since $K \# K^\star = K(0,0)$. Now $\mathcal{K} \in \mathrm{Fix}(\sigma_2)$, and with the specification that $U = T^\star$, the familly of knots

$$K\left(T^{\sigma_2}, U^{\sigma_2^{-1}}\right)$$

share the common Jones polynomial

$$V_{K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})} = \langle K \# K^\star \rangle.$$

By recycling the argument of theorem 5.3, we can reduce the comparison of the knots

$$K(T,U) \quad \text{and} \quad K\left(T^{\sigma_2}, U^{\sigma_2^{-1}}\right)$$

to the comparison of the HOMFLY polynomials

$$P_{K(0,0)} \quad \text{and} \quad P_{K(\sigma_2, \sigma_2^{-1})}.$$

As shown by the previous examples, this generates further pairs of distinct knots that are not Conway mutants despite sharing the same Jones polynomial.

The notable exception is the square knot, obtained from the connected sum of trefoil knots $3_1 \# 3_1^{\star}$. This is the connected sum of 2-bridge knots. It can be seen as $K(0,0)$ in the closure

$$K(T,U) = \left|(\sigma_1\sigma_2^{-1}\sigma_1)(\sigma_5^{-1}\sigma_4\sigma_5^{-1})\right|$$

but another view is given in figure 5.19.

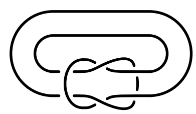


Figure 5.19: The square knot $3_1 \# 3_1^{\star}$.

From the diagram in figure 5.20 it can be seen that the action of σ_2 can cancel along a band connecting the tangles.

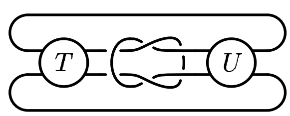
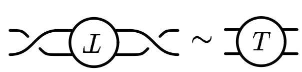


Figure 5.20: The knot $\left|(\sigma_1\sigma_2^{-1}\sigma_1)(\sigma_5^{-1}\sigma_4\sigma_5^{-1})\right|$.

This cancelation is of the form



so in the case that T is a rational tangle, their knot type is unaltered, while a more general tangle results in a Conway mutant of the original diagram. In particular, there is no change to the Jones polynomial.

In general however, the set of tangles $\mathcal{S}_2$ together with the set of 2-bridge knots (generated by B_3) provide a range of knots (and even links) having evaluation matricies contained in $\mathrm{Fix}(\sigma_2)$. In the cases discussed and the examples produced, we have seen that the HOMFLY polynomial may be used to distingush these knots. Thus, we conclude that this method of producing famillies knots sharing a common Jones polynomial is distinct from Conway mutation.

In the next chapter, the results of Eliahou, Kauffman and Thistlethwaite [9] will be restated using the braid actions introduced in this paper.

Chapter 6
Thistlethwaite Links

6.1 Construction

The group action of braids on tangles presented in this work was originally discussed by Eliahou, Kauffman and Thistlethwaite [9] in the course of study of the recently discovered links due to Thistlethwaite [33]. While it is still unknown whether there is a non trivial knot having Jones polynomial $V = 1$, Thistlethwaite's examples allow us to answer the question for links having more than 1 component.

Theorem (Thislethwaite). *For $n > 1$ there are non trivial n-component links having trivial Jones polynomial $V = \delta^{n-1}$.*

In the exploration of these links [9], it is shown that this is in fact a corollary of a much stronger statement.

Theorem (Eliahou, Kauffman, Thislethwaite). *For every n-component link L there is an infinite family of $(n+1)$-component links L' such that $V_{L'} = \delta V_L$.*

While these assertions are discussed at length in [9], the goal of this chapter is to present some of the examples in light of the group actions discussed in this work.

Definition 6.1. *A* Thislethwaite link *$H(T, U)$ is an external wiring of tangles $T, U \in \mathcal{S}_2$ modeled on the Hopf link.*

Our first task is to compute the evaluation matrix

$$\mathcal{H} = \begin{bmatrix} \langle H(0,0)\rangle & \langle H(0,\infty)\rangle \\ \langle H(\infty,0)\rangle & \langle H(\infty,\infty)\rangle \end{bmatrix}.$$

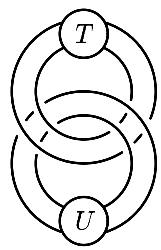


Figure 6.1: The Thislethwaite link $H(T, U)$

It is easy to see, by applications of the Reidemeister move R_2, that

$$\langle H(0,\infty)\rangle = \langle H(\infty,0)\rangle = \delta^2$$

and

$$\langle H(\infty,\infty)\rangle = \delta.$$

For the non trivial link $H(0,0)$, the computation of $\langle H(0,0)\rangle$ requires a little more work.

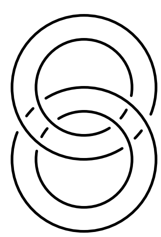
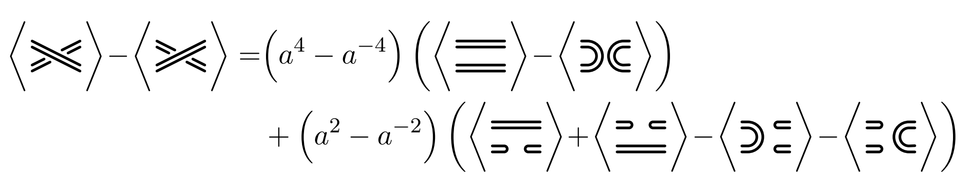


Figure 6.2: The link $H(0,0)$

For this, the following switching formula (stated in [20]) will be useful.

Switching Formula. *The equality*



holds for the Kauffman bracket, giving rise to equivalence in $\mathcal{TL}_4$.

PROOF OF THE SWITCHING FORMULA. First note that the double crossing  may be viewed as the braid $\sigma_2\sigma_1\sigma_3\sigma_2 \in B_4$. Thus, as

$$\sigma_i \longmapsto a + a^{-1}e_i$$

gives a representation of the braid group in $\mathcal{TL}_n$, we can represent this element of B_4 as

$$\begin{aligned}
&(a^2 + e_1 + e_2 + a^{-2}e_2e_1)(a^2 + e_2 + e_3 + a^{-2}e_3e_2) \\
=&a^4 + a^2e_2 + a^2e_3 + e_3e_2 \\
&+ a^2e_1 + e_1e_2 + e_1e_3 + a^{-2}e_1e_3e_2 \\
&+ a^2e_2 + e_2^2 + e_2e_3 + a^{-2}e_2e_3e_2 \\
&+ e_2e_1 + a^{-2}e_2e_1e_2 + a^{-2}e_2e_1e_3 + a^{-4}e_2e_1e_3e_2 \\
=&a^4 + a^2e_1 + (2a^{-2} + \delta + 2a^2)e_2 + a^2e_3 \\
&+ a^{-2}e_1e_3e_2 + a^{-2}e_2e_1e_3 + a^{-4}e_2e_1e_3e_2 \\
&+ e_1e_2 + e_1e_3 + e_2e_1 + e_2e_3 + e_3e_2.
\end{aligned}$$

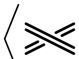
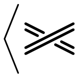
Similarly, the double crossing  may be viewed as $(\sigma_2\sigma_1\sigma_3\sigma_2)^{-1} \in B_4$ so that

$$\sigma_i^{-1} \longmapsto a^{-1} + ae_i$$

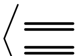
gives the representation

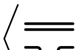
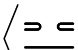
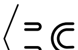
$$\begin{aligned}
&(a^{-2} + e_1 + e_2 + a^2e_2e_1)(a^{-2} + e_2 + e_3 + a^2e_3e_2) \\
=&a^{-4} + a^{-2}e_1 + (2a^{-2} + \delta + 2a^2)e_2 + a^{-2}e_3 \\
&+ a^2e_1e_3e_2 + a^2e_2e_1e_3 + a^4e_2e_1e_3e_2 \\
&+ e_1e_2 + e_1e_3 + e_2e_1 + e_2e_3 + e_3e_2.
\end{aligned}$$

Therefore

$\Big\langle$  $\Big\rangle - \Big\langle$  $\Big\rangle$

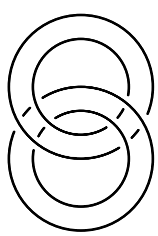
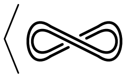
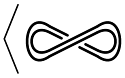
$$
\begin{aligned}
=&(a^2 + e_1 + e_2 + a^{-2}e_2e_1)(a^2 + e_2 + e_3 + a^{-2}e_3e_2) \\
&- (a^{-2} + e_1 + e_2 + a^2e_2e_1)(a^{-2} + e_2 + e_3 + a^2e_3e_2) \\
=&a^4 - a^{-4} + (a^2 - a^{-2})e_1 + (a^2 - a^{-2})e_3 \\
&+ (a^{-2} - a^2)e_1e_3e_2 + (a^{-2} - a^2)e_2e_1e_3 \\
&+ (a^{-4} - a^4)e_2e_1e_3e_2 \\
=&(a^4 - a^{-4})(1 - e_2e_1e_3e_2) \\
&+ (a^2 - a^{-2})(e_1 + e_3 - e_1e_3e_2 - e_2e_1e_3)
\end{aligned}
$$

$=\left(a^4 - a^{-4}\right)\Big(\Big\langle$  $\Big\rangle - \Big\langle$  $\Big\rangle\Big)$

$+\left(a^2 - a^{-2}\right)\Big(\Big\langle$  $\Big\rangle + \Big\langle$  $\Big\rangle - \Big\langle$  $\Big\rangle - \Big\langle$  $\Big\rangle\Big)$

as required. □

Applying the switching formula twice,

$\Big\langle$  $\Big\rangle = \delta^3 + \left(a^4 - a^{-4}\right)\Big(\Big\langle$  $\Big\rangle - \Big\langle$  $\Big\rangle\Big)$

$$
\begin{aligned}
&= \delta^3 + (a^4 - a^{-4}) \\
&\qquad \left[-(a^4 - a^{-4})(\delta - \delta^3) - (a^2 - a^{-2})(2 - 2\delta^2)\right] \\
&= \delta^3 + (a^4 - a^{-4})(\delta^2 - 1)\left[\delta(a^4 - a^{-4}) + 2(a^2 - a^{-2})\right] \\
&= \delta^3 + (a^4 - a^{-4})(\delta^2 - 1)\left[a^{-6} - a^{-2} + a^2 - a^6\right] \\
&= -a^{-14} - a^{-6} - 2a^{-2} - 2a^2 - a^6 - a^{14}
\end{aligned}
$$

so the evaluation matrix is

$$
\mathcal{H} = \begin{bmatrix} -a^{-14} - a^{-6} - 2a^{-2} - 2a^2 - a^6 - a^{14} & \delta^2 \\ \delta^2 & \delta \end{bmatrix}.
$$

Now consider the braid $\omega = \sigma_2^2\sigma_1^{-1}\sigma_2^2$

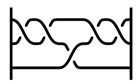


Figure 6.3: The braid $\omega \in B_3$

giving rise to the matrices

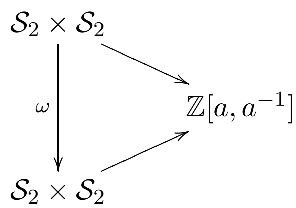
$$\Phi(\omega) = \begin{bmatrix} -a^{-1} + a^3 - a^7 & -a^{-11} + 2a^{-7} - 2a^{-3} + 2a - a^5 \\ a^{-3} & a^{-13} - a^{-9} + a^{-5} \end{bmatrix}$$

$$\Phi(\omega^{-1}) = \begin{bmatrix} -a^{-7} + a^{-3} - a & -a^{-5} + 2a^{-1} - 2a^3 + 2a^7 - a^{11} \\ a^3 & a^5 - a^9 + a^{13} \end{bmatrix}.$$

It can be checked (using MAPLE, for example) that

$$\Phi(\omega)\,\mathcal{H}\,\Phi(\omega^{-1})^\top = \mathcal{H}$$

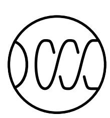
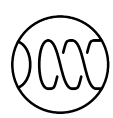
so that $\mathcal{H} \in \mathrm{Fix}(\omega)$, giving rise to a comutative diagram



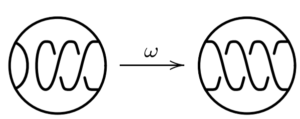
and

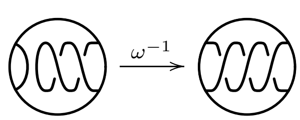
$$\langle H(T,U)\rangle = \langle H(T^\omega, U^{\omega^{-1}})\rangle.$$

As a first example, consider the tangles

$T =$  and $U =$ 

forming an unlink $H(T,U)$. Under the action of ω, we have





Therefore, the link $H(T^{\omega}, U^{\omega^{-1}})$ is two linked trefoils and, depending on orientation, $w(H(T^{\omega}, U^{\omega^{-1}})) = \pm 8$

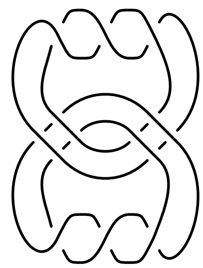


Figure 6.4: The link $H(T^{\omega}, U^{\omega^{-1}})$

so that

$$\langle H(T^{\omega}, U^{\omega^{-1}}) \rangle = \delta$$

and

$$V_{H(T^{\omega}, U^{\omega^{-1}})} = -a^{\pm 24}\delta.$$

However, the action of ω^2 leaves the writhe unchanged. This gives rise to a family of 2-component Thisltlethwaite links, all having Jones polynomial δ. Taking tangles T, U as in the previous example, the links

$$H(T^{\omega^{2n}}, U^{\omega^{-2n}})$$

have Jones polynomial δ for all $n \in \mathbb{Z}$. Moreover, for $n \neq 0$ the links obtained are non-trivial, since each component is the numerator closure of a tangle, giving rise to a pair of 2-bridge links that are geometrically essential to a pair of linked solid tori [32].

The first two links in this sequence (for $n = 1, 2$) are shown in figures 6.5 and 6.6.

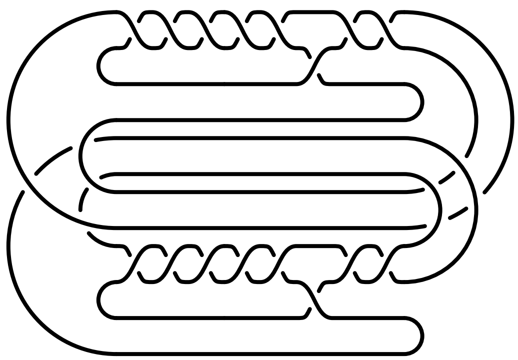


Figure 6.5: The result of ω^2 acting on $H(T, U)$

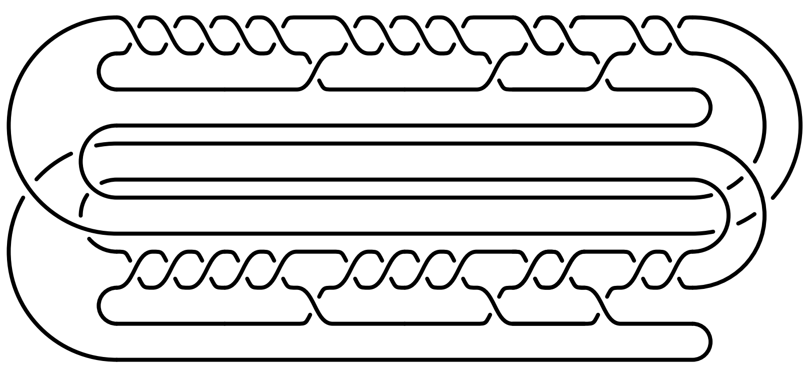
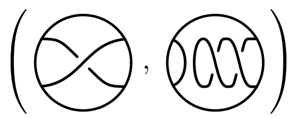


Figure 6.6: The result of ω^4 acting on $H(T, U)$

6.2 Some 2-component examples

Thistlethwaite's original discovery [33] consisted of links that had fewer crossings than those of the infinite sequence constructed above. Starting with the pair of tangles

$(T, U) =$ 

we obtain a trivial link $H(T, U)$ such that $w(H(T, U)) = -3$. Applying the action of ω to this link gives rise to a non-trivial link

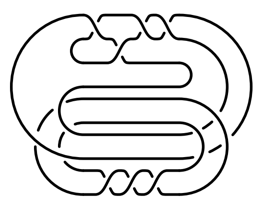


Figure 6.7: A non-trivial, 2-component link

such that

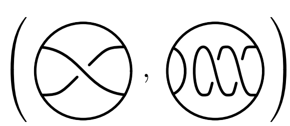
$$\langle H(T, U)\rangle = \langle H(T^{\omega}, U^{\omega^{-1}})\rangle$$

and

$$w\left(H(T^{\omega}, U^{\omega^{-1}})\right) = -3.$$

The result is a non-trivial link with trivial Jones polynomial δ.

Similarly, starting with the pair of tangles

$(T, U) =$ 

gives rise to another trivial link $H(T, U)$, in this case having $w(H(T, U)) = -1$. Applying the action of ω to this link gives rise to a non-trivial link

such that

$$\langle H(T, U)\rangle = \langle H(T^{\omega}, U^{\omega^{-1}})\rangle$$

and

$$w\left(H(T^{\omega}, U^{\omega^{-1}})\right) = -1.$$

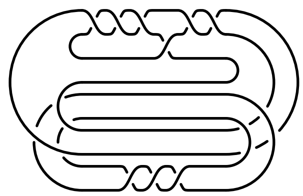


Figure 6.8: A non-trivial, 2-component link

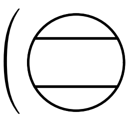
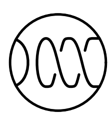
Again, the result is a non-trivial link with trivial Jones polynomial δ.

It has been shown that these examples are also members of an infinite family of distinct 2-component links having trivial Jones polynomials [9].

6.3 A 3-component example

It is possible to construct a 16-crossings non-trivial link with trivial Jones polynomial if we consider links of 3 components.

Starting with the pair of tangles

$(T, U) = \Big($  ,  $\Big)$,

gives a 3 component trivial link $H(T, U)$. In this case, $w(H(T, U)) = -2$ and applying the action of ω, the orientation of the resulting link may be chosen so that

$$w\left(H(T^{\omega}, U^{\omega^{-1}})\right) = -2$$

also. Thus, with this orientation,

$$V_{H(T^{\omega}, U^{\omega^{-1}})} = \delta^2.$$

In fact, with orientations chosen appropriately, this choice of tangles produces another infinite family of links

$$H(T^{\omega^n}, U^{\omega^{-n}})$$

for $n \in \mathbb{Z}$, each having trivial Jones polynomial [9].

The 16-crossing example is interesting, as it is may be constructed by linking two simple links: the Whitehead link (5_1^2), and the trefoil knot (3_1).

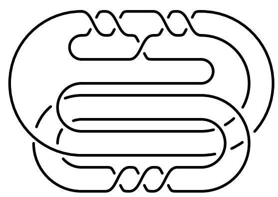


Figure 6.9: A non-trivial, 3-component link

6.4 Closing Remarks

While the search for an answer to question 3.2 continues, the method of mutation developed in this work provides a new tool in the pursuit of an example of a non-trivial knot having trivial Jones polynomial. Not only has this type of mutation produced Thistlethwaite's examples, it is also able to produce pairs of distinct knots sharing a common Jones polynomial that are not related by Conway mutation (theorem 5.3). In light of the fact that Conway mutation cannot alter an unknot so that it is knotted, it is desirable to have more general forms of mutation such as this braid action at our disposal.

We have produced pairs of knots sharing a common Jones polynomial. As these examples can be distinguished by their HOMFLY polynomials, they cannot be Conway mutants. In our development, it is shown that further such examples may be obtained either by altering the choice of tangles made, or by forming a special closure $|\beta|$ of a Kanenobu braid $\beta \in B_6$. In addition, it is shown that such a β may be produced from any given 3-braid.

It is hoped that further study of this new form of mutation will lead to a better understanding of the phenomenon of distinct knots sharing a common Jones polynomial. As well, it is possible that a better geometric understanding of this braid action could give rise to a better understanding of the Jones polynomial itself.

Bibliography

[1] J. W. Alexander. A lemma on systems of knotted curves. *Proc. Nat. Acad. Sci. USA*, 9:93–95, 1923.

[2] J. W. Alexander. Topological invariants of knots and links. *Trans. AMS*, 30(2):275–306, 1928.

[3] Emil Artin. Theorie der Zöpfe. *Abh. Math. Sem. Univ. Hamburg*, 4:47–72, 1925.

[4] Emil Artin. Theory of braids. *Ann. of Math.*, 48:101–126, 1947.

[5] Dror Bar-Natan. The knot atlas. www.math.toronto.edu/~drorbn.

[6] Joan S. Birman. *Braids, Links and Mapping Class Groups.* Number 82 in Annals of Mathematics Studies. Princeton University Press, Princeton, 1974.

[7] W. Burau. Über Zöpfgruppen und gleichsinnig verdrillte Verkettunger. *Abh. Math. Sem. Hanischen Univ.*, 11:171–178, 1936.

[8] J. H. Conway. An enumeration of knots and links. In J. Leech, editor, *Computational problems in abstract algebra*, pages 329–358. Pergamon Press, 1970.

[9] Shalom Eliahou, Louis Kauffman, and Morwen Thistlethwaite. Infinite families of links with trivial Jones polynomial. preprint.

[10] Roger Fenn. An elementary introduction to the theory of braids. Notes by Bernd Gemein.

[11] Roger Fenn. Hecke algebras, 2004. Notes from 'Knots in Vancouver'.

[12] P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millet, and A. Ocneanu. A new polynomial invariant of knots and links. *Bull. AMS*, 12:183–312, 1985.

[13] B. Hartley and T. O. Hawkes. *Rings, Modules and Linear Algebra.* Chapman and Hall Mathematics Series. Chapman and Hall, 1971.

[14] Jim Hoste and Jósef H. Przytycki. Tangle surgeries which preserve Jones-type polynomials. *Int. J. of Math.*, 8(8):1015–1027, 1997.

[15] Vaughan F. R. Jones. A polynomial invariant of knots via von Neumann algebras. *Bul. Amer. Math. Soc.*, 12:103–111, 1985.

[16] Vaughan F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math.*, 126:335–388, 1987.

[17] Taizo Kanenobu. Module d'Alexandre des noeuds fibré et polynôme de Hosokawa des lacements fibré. *Math. Sem. Notes Kobe Univ.*, 9:75–84, 1981.

[18] Taizo Kanenobu. Infinitely many knots with the same polynomial invariant. *Proc. AMS*, 97(1):158–162, 1986.

[19] Louis Kauffman. State models and the Jones polynomial. *Topology*, 26:395–407, 1987.

[20] Louis Kauffman and Sóstenes Lins. *Temperley-Lieb recoupling theory and invariants of 3-manifolds*. Number 134 in Annals of Mathematics Studies. Princeton University Press, Princeton, 1994.

[21] W. B. R. Lickorish. Three-manifolds and the Temperly-Lieb algebra. *Math. Ann.*, 290:657–670, 1991.

[22] W. B. R. Lickorish. Calculations with the Temperly-Lieb algebra. *Comment. Math. Helvetici.*, 67:571–591, 1992.

[23] W. B. Raymond Lickorish. *An Intoduction to Knot Theory*. Number 175 in Graduate Texts in Mathematics. Springer, 1991.

[24] A. A. Markov. Über die freie Äquivalenz geschlossener Zöpfe. *Rec. Math. Moscou*, 1:73–778, 1935.

[25] John W. Milnor. Infinite cyclic coverings. In John G. Hocking, editor, *Conference on the Topology of Manifolds*, volume 13 of *Complementary series in mathematics*, pages 115–133. Prindle, Weber and Schmidt, 1968.

[26] H. R. Morton. Treading knot diagrams. *Mth. Proc. Camb. Phil. Soc.*, 99:247–260, 1986.

[27] Kunio Murasugi and Bohdan I. Kurpita. *A Study of Braids*. Mathematics and Its Applications. Kluwer Academic Publishers, 1999.

[28] K. Reidemeister. Elementare Begründung der Knotentheorie. *Abh. Math. Sem. Univ. Hamburg*, 5:24–32, 1926.

[29] K. Reidemeister. *Knotentheorie*. Springer-Verlag, 1948.

[30] Dale Rolfsen. The quest for a knot with trivial Jones polynomial: Diagram surgery and the Temperley-Lieb algebra. In M.E. Bozhüyük, editor, *Topics in Knot Theory*, volume 399 of *Nato ASI Series C*, pages 195–210. Kluwer Academic Publishers, 1993.

[31] Dale Rolfsen. Gobal mutation of knots. *J. of Knot Theory*, 3(3):407–417, 1994.

[32] Dale Rolfsen. *Knots and Links*. AMS Chelsea, Providence, third edition, 2003.

[33] Morwen Thistlethwaite. Links with trivial Jones polynomial. preprint.

[34] Pierre Vogel. Representations of links by braids: a new algorithm. *Comment. Math. Helvetici*, 65:104–113, 1990.