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Knots with identical Khovanov homology

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We give a recipe for constructing families of distinct knots that have identical Khovanov homology and give examples of pairs of prime knots, as well as infinite families, with this property.

57M25, 57M27

1 Introduction

Khovanov homology is an invariant which associates a bi-graded abelian group to a knot (or link) in S^3 [7]. The Jones polynomial of the knot arises as a graded Euler characteristic of this theory, and as such questions about the Jones polynomial may be rephrased in terms of Khovanov homology. It is unknown for example, if either of these invariants detects the unknot. On the other hand, Khovanov homology is known to be strictly stronger than the Jones polyniomial: Bar-Natan provided examples of knots with the same Jones polynomial that are distinguished by Khovanov homology [2]. There have been many techniques developed for producing pairs of knots that have the same Jones polynomial, and it is natural to ask if these techniques also preserve Khovanov homology. One of the simplest techniques for generating distinct knots with the same Jones polynomial is mutation, however it is currently unknown if mutation of knots leaves Khovanov homology invariant [1, 19].

The aim of this note is to present a construction giving rise to distinct knots that cannot be distinguished using Khovanov homology. Our main tool is the long exact sequence in Khovanov homology which is presented, along with a review of Khovanov homology, in Section 2. In Section 3 we present a general construction for producing pairs of knots with identical Khovanov homology. This construction is applied in Section 4 to obtain pairs of distinct prime knots with identical Khovanov homology (Theorem 4.1). These examples are distinguished by the HOMFLYPT polynomial, and as such must have distinct triply-graded link homology [6, 8]. In Section 5 we give infinite families of distinct knots with identical Khovanov homology (Theorem 5.1), and in Section 6 we construct an infinite family of knots admitting a mutation that is not detected by Khovanov homology (Proposition 6.1). We conclude with a series of examples in Section 7.

2 Notation

We briefly review Khovanov homology to solidify notation, and refer the reader to Khovanov's original paper [7], as well as the work of Bar-Natan [2, 3] and Rasmussen [13].

The Khovanov complex of a knot K is generated by first considering an n-crossing diagram for K together with 2^n states, each of which is a collection of disjoint simple closed curves in the plane. Each state s is obtained from a choice of resolution (the 0-resolution) or) ((the 1-resolution) for each crossing. By fixing an order on the crossings, each state s may be represented by an n-tuple with entries in $\{0,1\}$ so that the states may be arranged at the vertices of the n-cube $[0,1]^n$ (the cube of resolutions for K). Let $|s|$ be the sum of the entries of the n-tuple associated to s (the height of s).

Let V be a free, graded $\mathbb{Z}$-module generated by $\langle v_-, v_+\rangle$, where $\deg(v_\pm) = \pm 1$. To each state we associate $V^{\otimes \ell_s}$ where $\ell_s > 0$ is the number of closed curves in the given state. The associated grading is referred to as the Jones grading, denoted by q. Set

$$\mathcal{C}^u(K) = \bigoplus_{u=|s|} V^{\otimes \ell_s}\{|s|\}$$

where $\{\cdot\}$ shifts the Jones grading via $(W\{j\})_q = W_{q-j}$. The chain groups of the Khovanov complex are given by

$$CKh^u_q(K) = \big(\mathcal{C}(K)[-n_-]\{n_+ - 2n_-\}\big)^u_q = \mathcal{C}^{u+n_-}_{q-n_+ +2n_-}(K)$$

where $[\cdot]$ shifts the homological grading u as shown. For a given orientation of K, $n_+ = n_+(K)$ is the number of positive crossings in K and $n_- = n_-(K)$ is the number of negative crossings in K.

The differentials $\partial^u : CKh^u(K) \to CKh^{u+1}(K)$ come from the collection of edges in the cube of resolutions moving from height u to height $u+1$. Each of these edges corresponds to exactly one of two operations ($m\colon\ V\otimes V \to V$ and $\Delta\colon\ V \to V\otimes V$) of a Frobenius algebra defined over V, since each edge can be identified with exactly one change of the form $\to$) (or) ($\to$. Fixing a convention so that the faces of the cube anti-commute, ∂^u is the sum of all the maps at the prescribed height. The

Khovanov homology $Kh(K)$, defined as the homology of the complex $(CKh^u(K), \partial^u)$, is an invariant of the knot K; the (unormalized) Jones polynomial of K arises as

$$\sum_j \sum_u (-1)^u q^j \dim(Kh^u_j(K) \otimes \mathbb{Q}).$$

Given a knot $K(\times)$ with a distinguished positive crossing, there is a short exact sequence

$$0 \longrightarrow \mathcal{C}\left(K(\asymp)\right)[1]\{1\} \longrightarrow \mathcal{C}\left(K(\times)\right) \longrightarrow \mathcal{C}\left(K()\,())\right) \longrightarrow 0.$$

Since $K()\,()$ inherits the orientation of $K(\times)$, we set $c = n_-\left(K(\asymp)\right) - n_-\left(K(\times)\right)$ for some choice of orientation on $K(\asymp)$ to obtain

$$0 \longrightarrow CKh^{u-c-1}_{q-3c-2}\left(K(\asymp)\right) \longrightarrow CKh^u_q\left(K(\times)\right) \longrightarrow CKh^u_{q-1}\left(K()\,())\right) \longrightarrow 0.$$

This short exact sequence gives rise to a long exact sequence

$$\to Kh^{u-c-1}_{q-3c-2}\left(K(\asymp)\right) \to Kh^u_q\left(K(\times)\right) \to Kh^u_{q-1}\left(K()\,())\right) \xrightarrow{\delta_*} Kh^{u-c}_{q-3c-2}\left(K(\asymp)\right) \to$$

where δ_* is the map induced on homology from (the component of) the differential $\delta : CKh^u_{q-1}\left(K()\,())\right) \to CKh^{u-c-1}_{q-3c-2}\left(K(\asymp)\right)$.

Similarly, for a knot $K(\times)$ with a distinguished negative crossing there is a long exact sequence:

$$\to Kh^u_{q+1}\left(K()\,())\right) \to Kh^u_q\left(K(\times)\right) \to Kh^{u-c}_{q-3c-1}\left(K(\asymp)\right) \xrightarrow{\delta_*} Kh^{u+1}_{q+1}\left(K()\,())\right) \to$$

We will make use of one further piece of structure on $Kh(K)$ introduced by Lee [9] and Rasmussen [12]. The bigraded abelian group $E_1^{u,q} = Kh^u_q(K) \otimes \mathbb{Q}$ associated to a knot K may be viewed as the first sheet of a spectral sequence (the Lee spectral sequence) with differential on the E_i sheet of bidegree $(1, 4i)$. In particular, each of the sums

$$\sum_{q \equiv 1 (\mathrm{mod}\, 4)} \sum_u (-1)^u \dim(E_i^{u,q})$$

and

$$\sum_{q \equiv -1 (\mathrm{mod}\, 4)} \sum_u (-1)^u \dim(E_i^{u,q})$$

must be constant for all i. Further, the spectral sequence converges to $E_\infty^{u,q} \cong \mathbb{Q} \oplus \mathbb{Q}$ with $E_\infty^{u,s\pm1} \cong \mathbb{Q}$, where the even integer s is Rasmussen's invariant. As a result the above constant is 1 in both cases, giving rise to:

$$\sum_{q \equiv 1 (\mathrm{mod}\, 4)} \sum_u (-1)^u \dim(Kh^u_q(K) \otimes \mathbb{Q}) = 1 \tag{1}$$

$$\sum_{q \equiv -1 (\mathrm{mod}\, 4)} \sum_u (-1)^u \dim(Kh^u_q(K) \otimes \mathbb{Q}) = 1 \tag{2}$$

3 Construction

Consider the knot $K = K_\beta(T, U)$ shown in Figure 1 (cf. [18]), where β is an element of the three strand braid group B_3 with inverse β^{-1}. T and U are tangles (or Conway tangles), that is $T = (B^3_T, \tau)$ and $U = (B^3_U, \mu)$ where B^3_T (respectively B^3_U) is a 3-ball containing a collection of arcs τ (respectively μ) that intersect the boundary of the 3-ball transversally in exactly 4 points [10, 15].

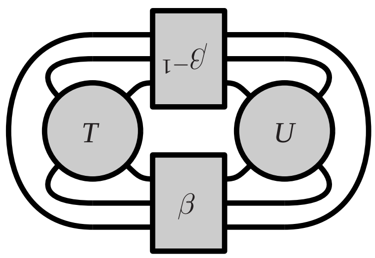
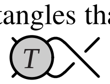
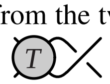
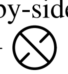
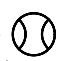
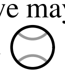


Figure 1: The knot $K_\beta(T, U)$.

There is a well defined $\mathbb{Z}$-action (a half-twist action) on the set of isotopy classes (fixing endpoints) of tangles that comes from the two strand braid group. For a given tangle T, write $T^\sigma =$  and $T^{\bar\sigma} =$  where $\langle\sigma\rangle = \mathbb{Z} \cong B_2$ and $\sigma\bar\sigma = e$ (that is, $\sigma =$  is the standard braid generator). Let $K^\sigma = K_\beta\left(T^\sigma, U^{\bar\sigma}\right)$.

The sum $T+U$ of two tangles is defined by side-by-side concatenation. This generalizes the half-twist-action: T^σ may be denoted $T +$  (σ *adds* a twist). A tangle T is called *simple* if $T +$  is isotopic (fixing endpoints) to . Note that T^σ is simple if and only if T is simple. Also, if T and U are simple, then so is the tangle $T + U$.

For the purposes of this paper, assume that the tangles considered have no closed components (that is, τ and μ are each a pair of arcs). Note also that since we are considering knots, we may restrict attention to tangles T and U that have connectivity of the form  and  (tangles having connectivity  generally give rise to links, moreover such tangles are never simple). Further, we assume throughout that the braid $\beta \in B_3$ is such that $K = K_\beta(T, U)$ has only one component.

Lemma 3.1 *For simple tangles T, U the knots $K = K_\beta(T, U)$ and $K^\sigma = K_\beta\left(T^\sigma, U^{\bar\sigma}\right)$ have identical Khovanov homology.*

Remark We could also consider a similar action that adds twists to tangles on the left; we make use of this in Section 6.

Remark It follows from Lemma 3.1 that the knots K and K^σ have identical Jones polynomial. This fact is proved in [18, Lemma 5.2] without the requirement that the tangles be simple.

Proof of Lemma 3.1 The proof is an application of the long exact sequences introduced in Section 2. The reader may find it useful to follow the general argument through on a particular example such as that of Section 7.1.

Our strategy is to distinguish two crossings of $K^\sigma = K_\beta\left(T^\sigma, U^{\bar\sigma}\right)$ (the two crossings added by the action of σ) and write $K(\times\times) = K_\beta\left(T^\sigma, U^{\bar\sigma}\right)$ so that $K(\asymp\asymp) = K_\beta(T,U)$. Since the tangles T and U are simple by hypothesis, it is easy to check that $K()\,(\times)$ and $K(\asymp)\,()$ (as well as $K(\times)\,()$ and $K()\,(\asymp)$) are diagrams for the 2-component trivial link (apply simplicity, and note that the braids are allowed to cancel). Let

$$L = Kh(\bigcirc\ \bigcirc) = V \otimes V \cong (\mathbb{Z})_{-2} \oplus (\mathbb{Z}\oplus\mathbb{Z})_0 \oplus (\mathbb{Z})_2$$

and note that $L^u = 0$ in all homological degrees $u \neq 0$.

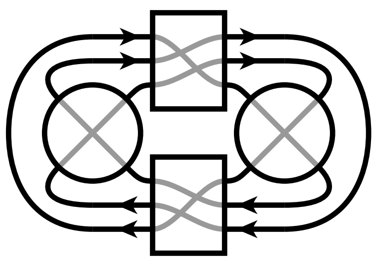


Figure 2: An orientation for the knot K^σ when the permutation associated to β is $(1\ 3\ 2)$.

Since it will be necessary to fix orientations, we divide into six cases according to the various possible permutations associated to braids in B_3. First suppose that the permutation associated to β is $(1\ 3\ 2)$ (as is the case for $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-2n}$, for example).

Now suppose that both T^σ and $U^{\bar\sigma}$ have connectivity of the form $\otimes$. Then it is easy to check that T and U have connectivity of the form $\ominus$. Moreover, since the permutation associated to β is $(1\ 3\ 2)$, the permutation associated to β^{-1} is $(1\ 2\ 3)$ and we can fix the orientation for $K(\times\times)$ shown in Figure 2 (note that it is clear from this diagram that the knot in question has one component). With this orientation in hand, we have that the distinguished crossing of T^σ is negative, while the distinguished crossing of $U^{\bar\sigma}$ is positive: $K(\times\times)$. Notice that the resolution $K(\asymp\times)$ (that is, the

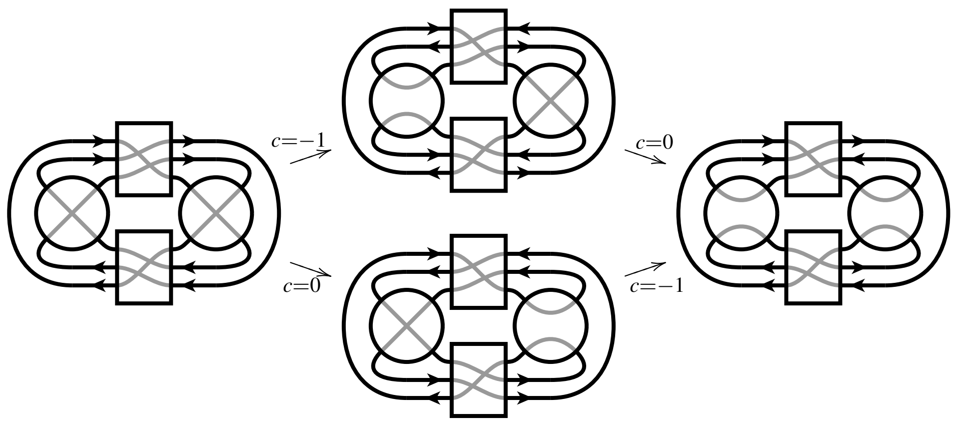


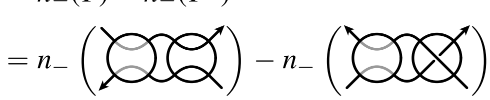
Figure 3: Values of c upon resolution.

tangle T) does not inherit this orientation. If we resolve with respect to the left-most (distinguished) crossing, we have the exact sequence

$$L^u_{q+1} \longrightarrow Kh^u_q K(\times\times) \longrightarrow Kh^{u-c}_{q-3c-1} K(\asymp\times) \longrightarrow L^{u+1}_{q+1}$$

where one can check that $c = n_-(K(\asymp\times)) - n_-(K(\times\times)) = -1$. Indeed, since the braids chosen are inverses of each other, the number of negative and positive crossings contributed by the braids remains constant. Therefore, to compute the values for c we need only consider the tangles T^σ and $U^{\bar\sigma}$. Upon resolution, notice that the orientation on $U^{\bar\sigma}$ is preserved (see Figure 3), while the new orientation for the resolution of T^σ (that is, T) has precisely one less negative crossing (the crossing we resolved). This is because the new orientation reverses the orientation on both strands (it can be checked that this will always preserve the number of positive and negative crossings) so that

$$\begin{aligned} c &= n_-\left(K(\asymp\times)\right) - n_-\left(K(\times\times)\right) \\ &= n_-(T) - n_-(T^\sigma) \end{aligned}$$



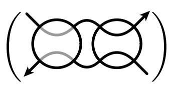
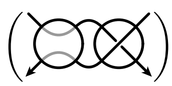
$$= -1.$$

Therefore, the exact sequence is given by

(3) $$L^u_{q+1} \longrightarrow Kh^u_q K(\times\times) \longrightarrow Kh^{u+1}_{q+2} K(\asymp\times) \longrightarrow L^{u+1}_{q+1}.$$

Now, resolving the second crossing, we make similar observations. The orientation on the strands of the tangle T are both reversed once more (see Figure 3), so that the number of negative crossings contributed by T is left unchanged. On the other hand, the resolution taking $U^{\bar\sigma}$ to U removes a positive crossing, and preserves the orientation on the tangle U. Therefore

$$\begin{aligned} c &= n_-\left(K(\asymp\asymp)\right) - n_-\left(K(\asymp\times)\right)\\ &= n_-(U) - n_-(U^{\bar\sigma})\\ &= n_-\left(\text{[diagram]}\right) - n_-\left(\text{[diagram]}\right)\\ &= 0\end{aligned}$$

and we have the exact sequence

$$L^{u-1}_{q-1} \longrightarrow Kh^{u-1}_{q-2}K(\asymp\asymp) \longrightarrow Kh^{u}_{q}K(\asymp\times) \longrightarrow L^{u}_{q-1}$$

which may be rewritten as

$$L^{u}_{q+1} \longrightarrow Kh^{u}_{q}K(\asymp\asymp) \longrightarrow Kh^{u+1}_{q+2}K(\asymp\times) \longrightarrow L^{u+1}_{q+1}. \tag{4}$$

Combining the exact sequences 3 and 4 we obtain the the diagram of exact sequences

$$\begin{array}{ccccccc} L^{u}_{q+1} & \longrightarrow & Kh^{u}_{q}K(\asymp\asymp) & \searrow & & & \\ L^{u}_{q+1} & \longrightarrow & Kh^{u}_{q}K(\times\times) & \longrightarrow & Kh^{u+1}_{q+2}K(\asymp\times) & \longrightarrow & L^{u+1}_{q+1} \\ & & & & & \searrow & L^{u+1}_{q+1} \end{array} \tag{5}$$

from which we deduce that

$$Kh^{u}_{q}K(\times\times) \cong Kh^{u+1}_{q+2}K(\asymp\times) \cong Kh^{u}_{q}K(\asymp\asymp)$$

for all homological gradings $u > 0$ since $L^u = 0$. Moreover, when $u = 0$ we have

$$\begin{array}{ccccccc} (\mathbb{Z})_{-2}\oplus(\mathbb{Z}\oplus\mathbb{Z})_0\oplus(\mathbb{Z})_2 & \longrightarrow & Kh^{0}_{q}K(\asymp\asymp) & \searrow & & & \\ (\mathbb{Z})_{-2}\oplus(\mathbb{Z}\oplus\mathbb{Z})_0\oplus(\mathbb{Z})_2 & \longrightarrow & Kh^{0}_{q}K(\times\times) & \longrightarrow & Kh^{1}_{q+2}K(\asymp\times) & \longrightarrow & 0 \\ & & & & & \searrow & 0 \end{array}$$

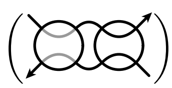
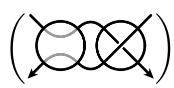
so that

$$Kh^{0}_{q}K(\times\times) \cong Kh^{0}_{q}K(\asymp\asymp)$$

for $q \neq -3, -1, 1$.

A slightly different diagram of exact sequences is obtained if the right-most distinguished crossing of $K(\times\times)$ is resolved first; we apply a similar argument for the (switched) values of c. This time, we first resolve the (positive) crossing of $U^{\bar{\sigma}}$ to obtain U with its orientation unchanged (see Figure 3). The induced orientation on T^{σ} reverses the orientation of both strands (as before) so that the number of positive and negative crossings contributed by T^{σ} are once more unchanged (of course, the contribution from the braids is constant, as before). Therefore, we lose only a positive crossing, and obtain

$$\begin{aligned} c &= n_-\left(K(\times\asymp)\right) - n_-\left(K(\times\times)\right) \\ &= n_-(U) - n_-(U^{\bar{\sigma}}) \end{aligned}$$

$= n_-$  $- n_-$ 

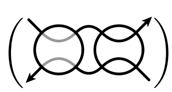
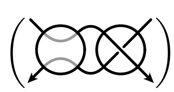
$= 0,$

which in turn gives the exact sequence

(6) $$L^{u-1}_{q-1} \longrightarrow Kh^{u-1}_{q-2}K(\times\asymp) \longrightarrow Kh^{u}_{q}K(\times\times) \longrightarrow L^{u}_{q-1}.$$

Resolving the distinguished crossing of T^{σ} to obtain T, we have that the orientation on U is once more preserved (see Figure 3), so that the contribution to c comes from comparing T and T^{σ} only. Once again, we remove the distinguished crossing (a negative crossing) and reverse orientation of both the strands of T. Therefore,

$$\begin{aligned} c &= n_-\left(K(\asymp\asymp)\right) - n_-\left(K(\times\asymp)\right) \\ &= n_-(T) - n_-(T^{\sigma}) \end{aligned}$$

$= n_-$  $- n_-$ 

$= -1$

and the we obtain the sequence

(7) $$L^{u-1}_{q-1} \longrightarrow Kh^{u-1}_{q-2}K(\times\asymp) \longrightarrow Kh^{u}_{q}K(\asymp\asymp) \longrightarrow L^{u}_{q-1}$$

where the gradings are shifted accordingly as in the case of the exact sequence 4. This time, sequences 6 and 7 combine to give the diagram of exact sequences

(8) $$\begin{array}{ccccccc} L^{u-1}_{q-1} & \searrow & & & & & \\ L^{u-1}_{q-1} & \longrightarrow & Kh^{u-1}_{q-2}K(\times\asymp) & \longrightarrow & Kh^{u}_{q}K(\times\times) & \longrightarrow & L^{u}_{q-1} \\ & & & \searrow & Kh^{u}_{q}K(\asymp\asymp) & \longrightarrow & L^{u}_{q-1} \end{array}$$

and we obtain the isomorphism

$$Kh^u_q K(\times\times) \cong Kh^{u-1}_{q-2} K(\times\asymp) \cong Kh^u_q K(\asymp\asymp)$$

whenever $u<0$. When $u=0$, we have

$$\begin{array}{ccccccc} 0 & \searrow & & & & & \\ 0 & \longrightarrow & Kh^{-1}_{q-2}K(\times\asymp) & \longrightarrow & Kh^0_q K(\times\times) & \longrightarrow & (\mathbb{Z})_{-2}\oplus(\mathbb{Z}\oplus\mathbb{Z})_0\oplus(\mathbb{Z})_2 \\ & & & \searrow & Kh^0_q K(\asymp\asymp) & \longrightarrow & (\mathbb{Z})_{-2}\oplus(\mathbb{Z}\oplus\mathbb{Z})_0\oplus(\mathbb{Z})_2 \end{array}$$

so that

$$Kh^0_q K(\times\times) \cong Kh^0_q K(\asymp\asymp)$$

for $q \neq -1,1,3$.

Combining the information from diagrams 5 and 8, we conclude that

$$Kh^u_q K(\times\times) \cong Kh^u_q K(\asymp\asymp)$$

except when $u=0$ and $q=\pm 1$. In fact, diagram 8 tells us that the torsion parts for $u=0$ and $q=\pm 1$ are isomorphic. Indeed, since L is torsion free we have

$$\begin{array}{ccccccc} 0 & \searrow & & & & & \\ 0 & \longrightarrow & \mathrm{Tor}\left(Kh^{-1}_{q-2}K(\times\asymp)\right) & \longrightarrow & \mathrm{Tor}\left(Kh^0_q K(\times\times)\right) & \longrightarrow & 0 \\ & & & \searrow & \mathrm{Tor}\left(Kh^0_q K(\asymp\asymp)\right) & \longrightarrow & 0 \end{array}$$

hence

$$\mathrm{Tor}\left(Kh^0_q K(\times\times)\right) \cong \mathrm{Tor}\left(Kh^0_q K(\asymp\asymp)\right)$$

for all q.

We pause here to remark that the cases with different connectivity for T^σ and $U^{\bar{\sigma}}$ ($(\asymp)$, $(\asymp)$ and $(\asymp)$, $\otimes$ and $\otimes$, $(\asymp)$) proceed in the same way, with only minor adjustments to the induced orientations. In fact, the proof amounts to reordering and/or rotating the oriented diagrams encountered in Figure 3. We leave this step to the reader.

To treat the other possible permutations associated to β it suffices to check that the same values for c are obtained upon resolution of the distinguished crossings. If this is the case, then the rest of the argument goes through unchanged. First notice that the if the permutation associated to β is (1 2 3) then it suffices to rotate each of the diagrams in Figure 3 by 180 degrees and exchange the diagrams of the middle column to obtain the same values for c. We do not need to consider the permutations (1) or (2 3) since it can be checked that if β has either of these associated permutations, the knot $K_\beta(T,U)$ will have more than one component. It remains to check the permutations (1 2) and (1 3). Notice that in either case, the oriented diagrams for $K(\times\times)$ and $K(\asymp\asymp)$ are exactly as in Figure 3 (with obvious adjustments to the connectivity of the braids). We leave it to the reader to check that the diagrams for $K(\asymp\times)$ and $K(\times\asymp)$ in both cases admit orientations that give rise to:

$$\begin{array}{ccccc} & & K(\asymp\times) & & \\ & \overset{c=-1}{\nearrow} & & \overset{c=0}{\searrow} & \\ K(\times\times) & & & & K(\asymp\asymp) \\ & \underset{c=0}{\searrow} & & \underset{c=-1}{\nearrow} & \\ & & K(\times\asymp) & & \end{array}$$

This in turn gives rise to the same diagrams of exact sequences 5 and 8 for all β such that $K_\beta(T,U)$ has one component.

Finally, note that acting by $\bar\sigma$ (instead of σ) on K switches the two exact sequences in each of the diagrams 5 and 8. This is due once more to the values obtained for c:

$$\begin{array}{ccccc} & & K(\asymp\times) & & \\ & \overset{c=0}{\nearrow} & & \overset{c=-1}{\searrow} & \\ K(\times\times) & & & & K(\asymp\asymp) \\ & \underset{c=-1}{\searrow} & & \underset{c=0}{\nearrow} & \\ & & K(\times\asymp) & & \end{array}$$

In particular, we obtain the same isomorphisms of Khovanov homology groups.

With these observations in hand, it remains now to analyze the the free part of $Kh^0_{\pm1}K(\times\times)$ and $Kh^0_{\pm1}(\asymp\asymp)$. To this end we work over $\mathbb{Q}$, and apply Equation 1 and Equation 2.

Suppose $q=1$, then from Equation 1 we have that

$$\sum_{q\equiv 1(\mathrm{mod}\,4)}\sum_u(-1)^u\dim(Kh^u_qK(\times\times)\otimes\mathbb{Q}) = \sum_{q\equiv 1(\mathrm{mod}\,4)}\sum_u(-1)^u\dim(Kh^u_qK(\asymp\asymp)\otimes\mathbb{Q})$$

and since all groups are isomorphic away from $(u,q)=(0,\pm1)$, this implies that

$$\dim(Kh^0_1K(\times\times)\otimes\mathbb{Q})=\dim(Kh^0_1K(\asymp\asymp)\otimes\mathbb{Q}).$$

In particular, $Kh^0_1K(\times\times)\otimes\mathbb{Q}\cong Kh^0_1K(\asymp\asymp)\otimes\mathbb{Q}$.

Applying a similar argument to the case $q = -1$ using Equation 2 gives the required isomorphism $Kh^0_{-1}K(\times\times) \otimes \mathbb{Q} \cong Kh^0_{-1}K(\asymp\asymp) \otimes \mathbb{Q}$ and we conclude that

$$Kh(K) \cong Kh(K^\sigma). \qquad \square$$

We have yet to see that the knots K and K^σ are distinct. This is the focus of Section 4, Section 5 and Section 6; examples are given in Section 7.

4 Distinct prime knots with identical Khovanov homology

According to Lickorish [10, Theorem 5], a tangle $T = (B^3_T, \tau)$ is *prime* if and only if the two-fold branched cover of B^3_T (branched over τ) is irreducible and boundary irreducible. Since the two-fold cover of a sphere with 4 branch points is a torus, prime tangles are those two-fold branched covered by non-trivial knot complements. Note that T is prime if and only if T^σ is prime.

Theorem 4.1 *For every simple, prime tangle T there exists a pair of distinct prime knots (each containing T) with identical Khovanov homology but distinct HOMFLYPT polynomial (and hence distinct triply-graded link homology).*

Proof Choose $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-2}$ and the pair of tangles $(T, T^\star)$ in the configuration of Figure 1 where $U = T^\star$ is the mirror image of T (hence prime and simple). This gives rise to a pair of knots $K_\beta(T, T^\star)$ and $K_\beta(T^\sigma, (T^\star)^{\bar\sigma})$ with identical Khovanov homology by applying Lemma 3.1. It is shown in [18, Theorem 1.1] that this pair of knots have distinct HOMFLYPT polynomials from which we conclude that the knots are distinct, and observe that they must have different triply-graded homology. Finally, it follows from the work of Lickorish [10, Theorem 1 and Lemma 2] that both of the knots constructed are prime whenever T is a prime tangle; a complete argument is given in [18, Theorem 1.1]. $\square$

Remark Since the pair of knots generated in the proof of Theorem 4.1 have distinct HOMFLYPT polynomial they can not be related by mutation (cf. Section 6).

Examples of knots arising as in Theorem 4.1 (in particular, examples of prime simple tangles) are given in Section 7.2.

5 Constructing infinite families

Although one needs a mechanism to prove that the knots obtained are distinct, the action of σ defined in Section 3 may be iterated to obtain infinite families of knots with identical Khovanov homology. Luse and Rong classified the particular familly $K_\beta(T,U)$ taking $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-2n}$ in the case where T and U horizontal full-twists [11].

Theorem 5.1 *For each $n\in\mathbb{N}$ there is an infinite family of distinct knots with identical Khovanov homology.*

Proof Fix $n\in\mathbb{N}$ and consider the family of knots $K_\ell = K_\beta(\sigma^{2\ell},\sigma^{-2\ell})$ where $\sigma^{2\ell}$ is the tangle consisting of ℓ horizontal full-twists, and $\beta=\sigma_1^{-1}\sigma_2\sigma_1^{-2n}$. The tangles are clearly simple, so by iterating Lemma 3.1 K_ℓ and $K_{\ell'}$ have identical Khovanov homology for any $\ell,\ell'\in\mathbb{Z}$. According to [11, Theorem 1.1], K_ℓ and $K_{\ell'}$ are distinct knots whenever $\gcd(\ell,2n+1)\neq\gcd(\ell',2n+1)$. If $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime decomposition of $2n+1$, we can choose $\ell=p_i$ (for any of the $i\in\{1,\ldots,k\}$) so that $\gcd(\ell,2n+1)=p_i$. Letting ℓ' range over all primes that do not appear in the prime decomposition of $2n+1$ gives the result. □

Remark The classification of this family of knots in the case $n=1$ is due to Kanenobu [5]; this example is given in Section 7.4.

6 A remark on mutation

Knot mutation (cf. [15]) is a well known operation on tangles that alters knots without changing any of the skein-type polynomial invariants (ie. Jones, HOMFLYPT, ...). Although Wehrli has given examples of split links related by mutation that have different Khovanov homology [19], it is unknown if mutation preserves Khovanov homology for knots [1, 19]. As a third application of Lemma 3.1, we give an infinite family of knots that admit a mutation which is not detected by Khovanov homology.

Consider the familly of knots $K_m(T) = K_\beta(\sigma^m,T)$ as in Figure 1, where σ^m is the tangle consisting of $m\in\mathbb{Z}$ horizontal half-twists (and β is such that $K_m(T)$ has only one component).

Proposition 6.1 *The mutation μ that flips a simple tangle T in the knot digram $K_m(T)$ across the horizontal axis is not detected by Khovanov homology. That is $Kh(K_m(T))\cong Kh(K_m(\mu T))$ for all $m\in\mathbb{Z}$.*

Proof If we consider a similar construction to that of Section 3 allowing B_2 to act on the left (ie. $\sigma\colon T \mapsto$  and $\bar{\sigma}\colon T \mapsto$ ), the proof of Lemma 3.1 goes through in the same way for this left action (on the same class of knots), and hence leaves Khovanov homology invariant. Indeed, one need only consider a 180 degree rotation of Figure 1, and the proof goes through verbatim after renaming $\beta = \beta^{-1}$, $T = \cap$ and $U = \perp$.

Since the tangles T and σ^m are simple, we can can apply Lemma 3.1 to $K_m(T)$ resulting in a new knot $K_{m+1}(T^{\bar{\sigma}})$ with identical Khovanov homology (note that with this notation $(\sigma^m)^\sigma = \sigma^{m+1}$). The key observation is that acting on the left by the inverse $\bar{\sigma}$ removes the twist added to σ^m, while the result of acting by σ on the left of $T^{\bar{\sigma}}$ is a tangle isotopic (fixing endpoints) to the mutant μT. Hence we obtain a third knot $K_m(\mu T)$ – precisely the desired mutant – with identical Khovanov homology. □

The elements of the proof of Proposition 6.1 can be seen in the example given in Section 7.5, in particular Figure 10.

7 Examples

The preceding sections show that the construction of Section 3 gives rise to a wide range of knots; in this section we give some particular examples. The notation for knots used below is consistent with Rolfsen's notation [14] for knots with fewer that 11 crossings, and KNOTSCAPE notation [17] otherwise up to mirrors (see also the Knot Atlas [4]).

7.1 A first example

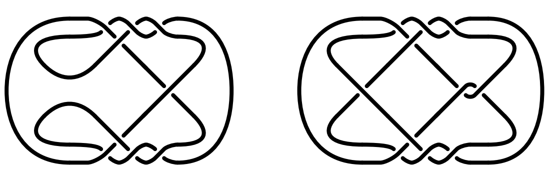


Figure 4: The knots 8_8 and 10_{129}.

The knots 8_8 and 10_{129} admit diagrams of the form of Figure 1 with $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-2}$, as shown in Figure 4. This explains the coincidence in Khovanov homology enjoyed by this pair of knots, a fact well documented in [4]. It may be illustrative to revisit the proof of Lemma 3.1 with this particular example in hand. Observe that as a result of this choice of diagrams, 8_8 is obtained by resolving the distinguished crossings shown in Figure 5 by $\to$ and $\to$.

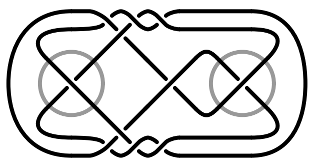


Figure 5: The distinguished crossings for the knot 10_{129}.

The knot $K( ) = 10_{137}$, so that the diagram of exact sequences 5 gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & Kh^u_q(8_8) & & & & \\ & & & \searrow & & & \\ 0 & \longrightarrow & Kh^u_q(10_{129}) & \longrightarrow & Kh^{u+1}_{q+2}(10_{137}) & \longrightarrow & 0 \\ & & & & & \searrow & \\ & & & & & & 0 \end{array}$$

for $u > 0$ and hence the isomorphism $Kh^u_q(8_8) \cong Kh^{u+1}_{q+2}(10_{137}) \cong Kh^u_q(10_{129})$ for $u > 0$. Similarly, The knot $K( ) = 8_9$ (this knot is shown in Figure 9), so that the diagram of exact sequences 8 gives

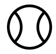
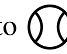
$$\begin{array}{ccccccc} 0 & & & & & & \\ & \searrow & & & & & \\ 0 & \longrightarrow & Kh^{u-1}_{q-2}(8_9) & \longrightarrow & Kh^u_q(10_{129}) & \longrightarrow & 0 \\ & & & \searrow & & & \\ & & & & Kh^u_q(8_8) & \longrightarrow & 0 \end{array}$$

for $u < 0$ and $Kh^u_q(8_8) \cong Kh^{u+1}_{q+2}(8_9) \cong Kh^u_q(10_{129})$ for $u < 0$.

As in the proof of Lemma 3.1, the isomorphism in homological grading $u = 0$ follows from Diagram 8 (for the torsion part) and Equation 1 and Equation 2 (for the free part).

Remark Note that this example illustrates the case of tangle connectivity of the form for T and for U in the proof of Lemma 3.1.

7.2 Pairs of non-mutant prime knots

To illustrate Theorem 4.1, we first need examples of prime simple tangles; these are provided in Figure 6. The fact that these are prime tangles for all $k \geq 0$ is shown by Lickorish [10, Section 2 Example (a)]. That these tangles are simple for $k \geq 0$ is an application of $k+2$ Reidemeister type II moves followed by a single Reidemeister type I move to see that $T +$  is isotopic (fixing endpoints) to .

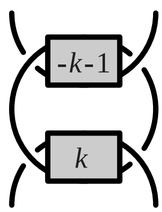


Figure 6: A prime simple tangle for $k \geq 0$.

The knots obtained from this construction in the case $k = 0$ are shown in Figure 7.

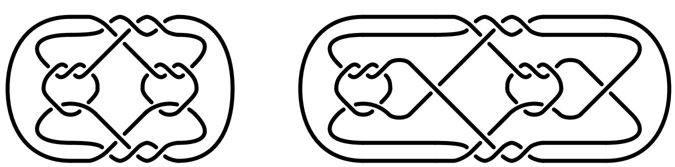
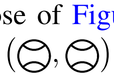
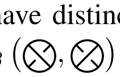
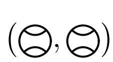
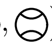


Figure 7: Non-mutant prime knots with identical Khovanov homology.

Remark In the proof of Theorem 4.1, the fact that the knots obtained (for example, those of Figure 7) have distinct HOMFLYPT polynomials depends on the fact that K_β  and K_β  have distinct HOMFLYPT polynomial (cf. [18, Theorem 1.1]). In the examples constructed for the proof Theorem 4.1 using $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-2}$, the knots are $4_1 \# 4_1$ and 8_9; these have distinct HOMFLYPT polynomials (cf. Section 7.4). However any such pair will do, and many more examples exist; we give two to conclude this section.

If $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-3}$ then $K = K_\beta$  $= 5_2 \# 5_2^\star$ and we obtain $K^\sigma = 10_{48}$. These have the same Khovanov homology by Lemma 3.1, while K and K^σ are distinguished by the HOMFLYPT polynomial.

If $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-2}$ then $K = K_\beta$ (, ) $= 6_3\#6_3$ and we obtain $K^\sigma = 12^a_{819}$. Again, these have the same Khovanov homology, while K and K^σ are distinguished by the HOMFLYPT polynomial.

Remark The reader may have observed that the base case in all of these examples is provided by taking a connected sum of a 2-bridge knot with its mirror image. While it is tempting to guess that any such connected sum will give rise to a family of examples, we leave it as an exercise to show that (at very least) the $(2,n)$-torus knots should be omitted since the action of σ is trivial on these examples (consider $\beta = \sigma_1^n$ or $\beta = \sigma_1^{-1}\sigma_2^{n-2}\sigma_1^{-1}$).

7.3 An aside on non-simple tangles

It is natural to ask if Lemma 3.1 holds without the simplicity assumption on the tangles. For example, the knots 12^a_{990} and 12^a_{1225} arise in this way (consider the braid closure of $\beta\sigma_2^3\beta^{-1}\sigma_2^{-3}$ where $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-2}$). A second example of this phenomenon is given by the knots 12^a_{427} and 15^n_{45009} shown in Figure 8. Both of these examples share the same Khovanov homology (verified using the software `KhoHo` [16]). Indeed, as noted in Section 3, the simplicity requirement on the tangles is not required to show that knots obtained in this way have identical Jones polynomial [18]. However it seems optimistic (though tempting) to conjecture that Lemma 3.1 holds for all tangles.

Question Is there a knot $K_\beta(T,U)$ with non-simple tangles for which the action of σ is detected by Khovanov homology?

While the construction of Section 3 provides a wide range of examples of knots with identical Khovanov homology, the examples given in this section serve as a reminder that the restriction to simple tangles is a particularly special case.

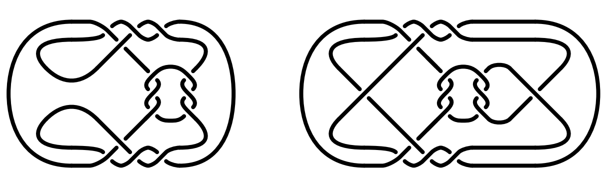


Figure 8: The knots 12^a_{427} and 15^n_{45009}.

7.4 Kanenobu's knots

It has been shown that the action of B_3 may be iterated to obtain infinite families. We illustrate the case $n = 1$ of Theorem 5.1 so that the braid in question is once more $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-2}$ to obtain a particular infinite family of knots with identical Khovanov homology. Let $K = K_\beta\left(, \right)$ so that $K^\sigma = K_\beta\left(, \right)$; the knots $K = 4_1\#4_1$, $K^\sigma = 8_9$ and $K^{\sigma^2} = 12^n_{462}$ are shown in Figure 9. By Lemma 3.1 these knots have the same Khovanov homology, while $4_1\#4_1$ and 8_9 (equivalently 8_9 and 12^n_{462}) have different HOMFLYPT polynomials. It should be noted that $4_1\#4_1$ and 12^n_{462} share the same HOMFLYPT polynomial, and the interested reader should consult [5] in which Kanenobu originally classified this example. In particular, this provides an infinite family of distinct knots with homology $Kh(8_9)$, and the $n = 1$ case for Theorem 5.1.

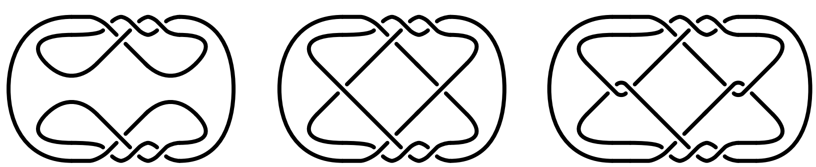


Figure 9: The first three knots in Kanenobu's sequence.

7.5 Mutants

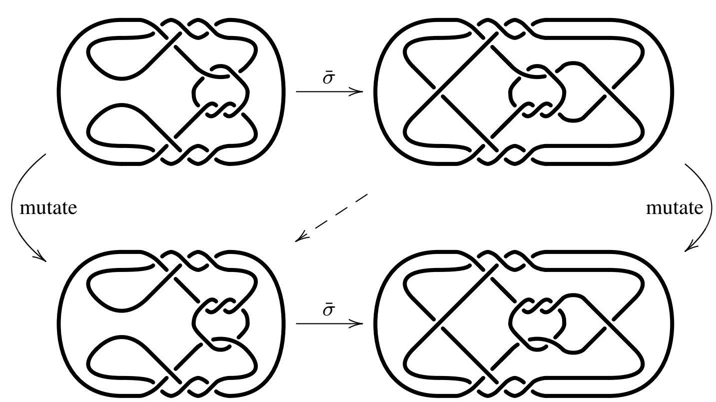


Figure 10: Two pairs of mutants illustrating Proposition 6.1.

The family of non-alternating knots with 13 crossings given in Figure 10 illustrate the construction used in the proof of Proposition 6.1. The four knots in question are arranged in Figure 10 so that pair in the first row (13^n_{164} and 13^n_{922}) are related by twisting (and hence have identical Khovanov homology by Lemma 3.1), as are the pair in the second row (13^n_{161} and 13^n_{795}). Note that we are acting by the inverse $\bar{\sigma}$ in this example. The columns are related by mutation (flipping across the horizontal axis), and the diagonal arrow in Figure 10 corresponds to the left action of $\sigma \in B_2$ used in the proof of Proposition 6.1, the second step of the mutation relating 13^n_{164} and 13^n_{161} (the knots in the left column of Figure 10). That is, each mutant pair (and indeed, any mutant pair of form given in the proof of Proposition 6.1) can be seen as the composition of right (twist) action, followed by a left (untwist) action.

Acknowledgements

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