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Involutions sur les variétés de dimension trois et homologie de Khovanov

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LIAM WATSON

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To the memory of my grandfather, James Watson.

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This work grew out of an interest in better understanding Khovanov homology. Not too long after suggesting that this might be a reasonable area for doctoral research, Steve Boyer asked “what does Khovanov homology have to say about surgery?” I would like to think of this work as the beginning of an answer to his question, and I thank him for answering endless questions pertaining to Dehn surgery, geometric topology, and the exceptional surgery problem, as well as many enlightening discussions and continued support.

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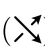
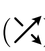
GLOSSARY OF SYMBOLS

$\cong$	homeomorphic, isomorphic
$\simeq$	equivalence of knots and links
$\hookrightarrow$	proper embedding
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{F}$	natural numbers, integers, rationals, reals, the two-element field
$\lfloor \cdot \rfloor$	the floor function
$\lceil \cdot \rceil$	the ceiling function
$[a_1, \ldots, a_r]$	a continued fraction of length r
B_3	the three-strand braid group
S^2, RP^2	the 2-sphere, real projective plane
B^3, S^3	the 3-ball, 3-sphere
$K^\star$	the mirror image of the knot K
$-Y$	the manifold Y with orientation reversed
$S^3_{p/q}(K)$	the result of $\frac{p}{q}$ Dehn surgery on a knot K
$\Delta(\alpha, \beta)$	distance between slopes α, β
$M(\alpha)$	the result of Dehn filling M along a slope α
$\tau(n), \tau(\frac{p}{q})$	a branch set associated to integer, or rational, Dehn filling
$\mathbf{\Sigma}(S^3, L)$	the two-fold branched cover of S^3, branched over a link L
$\mathbf{\Sigma}(X, Y)$	the two-fold branched cover of X, branched over $Y \hookrightarrow X$, whenever this makes sense
$(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$	a genus g Heegaard decomposition
$(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$	a genus g pointed Heegaard decomposition
λ_M	the rational longitude
$H_n(X; R)$	the n^{th} (singular) homology of X with coefficients in R

$H^n(X;R)$	the n^{th} cohomology of X with coefficients in R
$\Delta_L(t)$	the Alexander polynomial of a link L
$V_L(t)$	the Jones polynomial of a link L
$\widehat{V}_L(t)$	the unnormalized Jones polynomial of a link L
$\sigma(L)$	the signature of a link L
$\det(L)$	the determinant of a link L
$g(K)$	the Seifert genus of a knot K
$w(L)$	the width of a link L
$n_+(L)$	number of positive () crossings for an (oriented) link L
$n_-(L)$	number of negative () crossings for an (oriented) link L
CKh	the Khovanov complex
Kh	Khovanov homology
$\widetilde{\mathrm{CKh}}$	the reduced Khovanov complex
$\widetilde{\mathrm{Kh}}$	reduced Khovanov homology
$[\cdot,\cdot]$	shift operator (for bigradings)
$[\cdot]$	shift operator (for single gradings)
$\widetilde{\mathrm{Kh}}_\sigma$	σ-normalized Khovanov homology
$H_*(A \to B)$	the homology of the mapping cone of $A \to B$
Supp	the support of reduced Khovanov homology in the primary grading
$\mathrm{Sym}^g\,\Sigma_g$	the g-fold symmetric product
$\widehat{\mathrm{HF}}$	the "hat" version of Heegaard-Floer homology
$\mathrm{HF}^-, \mathrm{HF}^+, \mathrm{HF}^\infty$	the "minus", "plus", "infinity" versions of Heegaard-Floer homology
$\mathrm{HF}_{\mathrm{red}}$	reduced Heegaard-Floer homology
$\widehat{\mathrm{HFK}}$	knot Floer homology

RESUMÉ

Cette thèse établit, et étudie, un lien entre l'homologie de Khovanov et la topologie des revêtements ramifiés doubles. Nous y introduisons certaines propriétés de stabilité en homologie de Khovanov, dont nous dérivons par la suite des obstructions à l'existence de certaines chirurgies exceptionnelles sur les nœuds admettant une involution appropriée. Ce comportement, analogue à celui de l'homologie de Heegaard-Floer sous chirurgie, renforce ainsi le lien existant (dû à Ozsváth et Szabó) entre homologie de Khovanov, et homologie d'Heegaard-Floer des revêtements ramifiés doubles. Dans l'optique de poursuivre et d'exploiter plus avant cette relation, les méthodes développées dans ce travail sont appliquées à l'étude des L-espaces, et à déterminer, en premier lieu, si l'homologie de Khovanov fournit un invariant des revêtements ramifiés doubles, et en deuxième lieu, si l'homologie de Khovanov permet de détecter le nœud trivial.

ABSTRACT

This thesis establishes and investigates a relationship between Khovanov homology and the topology of two-fold branched covers. Stability properties for Khovanov homology are introduced, and as a result we obtain obstructions to certain exceptional surgeries on knots admitting an appropriate involution. This stable behaviour is analogous to the behaviour of Heegaard-Floer homology under surgery, strengthening the relationship (due to Ozsváth and Szabó) between Khovanov homology and the Heegaard-Floer homology of the two-fold branched cover. In the interest of pursuing and exploiting this relationship further, the methods developed in this work are applied to the study of L-spaces, as well as to the questions of whether Khovanov homology yields an invariant of two-fold branched covers and whether Khovanov homology detects the trivial knot.

INTRODUCTION

Khovanov's introduction of a homology theory for links in the three-sphere (Khovanov, 2000), followed by Bar-Natan's early work providing calculations of this invariant for knots with 11 crossings (Bar-Natan, 2002), immediately pointed to phenomenon demanding explanation. While these calculations exhibited that Khovanov homology was strictly stronger than the Jones polynomial (Jones, 1985) – a quantity arising as graded Euler characteristic – a vast majority of these small knots had homology that could be determined from the Jones polynomial and the signature. In particular, many of the knots in question had homology supported in a single diagonal, and such knots became referred to as thin.

It was subsequently conjectured that any non-split alternating link should be thin, and this was later proved in the seminal work of Lee (Lee, 2005). The machinery developed in the proof of this fact led to the definition of the Lee-Rasmussen spectral sequence, and ultimately Rasmussen's definition of the s invariant together with his celebrated combinatorial proof of the Milnor conjecture (Rasmussen, 2004a). It was immediately clear that Khovanov homology contained powerful geometric information, while being highly computable by virtue of its combinatorial definition.

The question of homological width more generally, that is, the number of diagonals supporting the Khovanov homology of a given link, has received continued attention. In particular, Shumakovitch provided further computations and conjectures in his work (Shumakovitch, 2004b), while Turner showed that torus knots provide examples of arbitrarily wide homology (Turner, 2008) (see also (Stošić, 2007)). Ozsváth and Manolescu extended Lee's result by exhibiting that quasi-alternating links (a class strictly larger that alternating) are thin (Manolescu and Ozsváth, 2007), and more recently, Lowrence has studied the width of closures of 3-braids (Lowrance, 2009).

In the direction of applications, Ng showed that Khovanov homology yields bounds on the Thurston-Benequin number of a knot (Ng, 2005), while Plamenevskaya defined a transverse invariant from Khovanov homology (Plamenevskaya, 2006a), related to the contact invariant in Heegaard-Floer homology (Plamenevskaya, 2006b) (see also (Baldwin and Plamanevskaya, 2008)). Both results point to interesting interaction with contact topology.

However despite these applications Khovanov homology, like the Jones polynomial, still lacks a complete geometric understanding. Various programs and frameworks exist in pursuit of this important open problem.

For example, Seidel and Smith have defined a homology theory for links from symplectic geometry, conjectured to be equal to a suitable grading-collapsed version of Khovanov homology (Seidel and Smith, 2006). Further, it has been observed that there is a coincidence between the homology of an $SU(2)$ representation space of the fundamental group of the knot complement, and Khovanov homology (for certain simple knots). Two frameworks for studying this phenomenon have been proposed by Kronheimer and Mrowka (Kronheimer and Mrowka, 2008) and Jacobsson and Rubinsztein (Jacobsson and Rubinsztein, 2008). In addition, work of Gukov et. al. proposes a conjectural relationship between generalizations of Khovanov homology (due to Khovanov and Rozansky (Khovanov and Rozansky, 2008)), and BPS invariants related to string theory, actively studied via Gromov-Witten theory (see for example (Gukov et al., 2007; Gukov et al., 2005)).

With this in mind, further geometric applications of the theory should be pursued. Such a pursuit should shed light on the geometric underpinnings of Khovanov homology, while developing the theory's role in low-dimensional topology and exploiting the combinatorial nature of the theory.

That further applications should exist follows from an important advance due to Ozváth and Szabó: Khovanov homology may be viewed as the E^2 term of a spectral sequence converging to the Heegaard-Floer homology of the two-fold branched cover (Ozsváth and

Szabó, 2005c). Since the latter has seen many powerful applications in low-dimensional topology since its inception, it seems reasonable to hope that Khovanov homology – viewed as an approximation of Heegaard-Floer homology in this setting – might hold further geometric information about two-fold branched covers. Indeed, it was an interest in better understanding the higher terms and differentials of this spectral sequence that led to many of the results in this thesis.

Summary of principal results

The primary results in this thesis may be broken into three parts.

Homological width as a surgery obstruction. Lee's result, combined with work of Hodgson and Rubinstein (Hodgson and Rubinstein, 1985) imply that if $\mathbf{\Sigma}(S^3, L)$ is a lens space then L must be a thin link, where $\mathbf{\Sigma}(S^3, L)$ denotes the two-fold branched cover of the three-sphere with branch set L (see Theorem 4.24). Following work of Montesinos (Montesinos, 1976), as well as work of Boileau and Otal (Boileau and Otal, 1991), we show (see Theorem 4.25):

Theorem. *If $\mathbf{\Sigma}(S^3, L)$ has finite fundamental group then L is supported in at most two diagonals.*

Thus, relaxing Lens spaces to manifolds with finite fundamental group, the homological width of the associated branch sets remains relatively tame.

Given a strongly invertible knot in S^3, Dehn surgery on K may be viewed as a branch cover $S^3_{p/q}(K) \cong \mathbf{\Sigma}(S^3, \tau(\frac{p}{q}))$. The width of the branch set is well behaved, as a result of a stability lemma (see Lemma 5.1) established in Chapter 5. This in turn implies that the quantity $w_{\min}$ (respectively $w_{\max}$), the minimum (respectively maximum) width attained by the branch sets $\tau(n)$ corresponding to integer fillings is well defined. In fact these quantities typically give upper and lower bounds for the width of the link $\tau(\frac{p}{q})$ (see Theorem 6.5):

Theorem. *Let K be a strongly invertible knot in S^3, so that $S^3_{p/q}(K) \cong \mathbf{\Sigma}(S^3, \tau(\frac{p}{q}))$.*

Under mild genericity conditions,[1] $w_{\min} > 2$ *implies that Dehn surgery on* K *never yields a manifold with finite fundamental group. Moreover,* $w_{\min}$ *may be determined on a finite collection on integers.*

Note that $w_{\min} > 1$ may be applied as an obstruction to lens space surgeries in the same way (see Theorem 6.4). While the genericity assumptions we impose seem mild relative to the branch sets that arise in practice, we remark that in the broader context of the exceptional surgery problem, tools such as the cyclic surgery theorem (Culler et al., 1987), as well as extensions due to Boyer and Zhang, (Boyer and Zhang, 1996; Boyer and Zhang, 2001) may be used to restrict the cases that must be checked should non-generic phenomena be encountered.

Khovanov homology and two-fold branched covers. By studying constructions of branch sets for Seifert fibered spaces, we answer a question of P. Ozsváth (see Corollary 7.5):

Theorem. *The total rank of the reduced Khovanov homology is not an invariant of the two-fold branched cover.*

This is demonstrated by example: Brieskorn spheres arise as two-fold branched covers of S^3, typically in two distinct ways. We determine these branch sets, and establish that the rank of the Khovanov homology distinguishes the pair of branch sets in some cases (see Example 7.4).

As discussed by Ozsváth, this question arises naturally when considering the possibility of defining an extension of Khovanov's invariant for more general closed 3-manifolds, by specifying the E^2 term of a spectral sequence converging to any theory satisfying Floer's exact triangle (Ozsváth, 2008). Such a generalization should coincide with Khovanov homology when restricting to two-fold branched covers.

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[1] Easily verified, and seemingly always satisfied, these conditions are disussed at length (and in particular made precise), in Section 5.6 and thoughout Chapter 6.

Invariants for detecting the trivial knot. In light of the spectral sequence relating Khovanov homology and Heegaard-Floer homology, one might hope to gain information about the former by applying geometric properties of the latter. In particular, the following open problem is of considerable interest:

Question. *Does Khovanov homology detect the trivial knot?*

This does not follow immediately from the spectral sequence, due to the existence of manifolds with Heegaard-Floer homology of rank one. However, such manifolds are rare, and as a result the above question has an affirmative answer on a particularly large class of knots (see Theorem 8.2). As a result, by pre-composing with certain satellite constructions, it is possible to construct a combinatorial invariant that detects the trivial knot using Khovanov homology (see Corollary 8.4):

Theorem. *The Khovanov homology of the* $(2,1)$*-cable of a knot detects the trivial knot.*

This result is joint work with M. Hedden (Hedden and Watson, 2008), and is in fact a single example of a satellite construction with which to pre-compose to yield an invariant for detecting the trivial knot.

Overview

The first three chapters of this work comprise an idiosyncratic introduction to the areas in which this work is cast; the majority of the content can be found elsewhere, and we endeavour to provide thorough references as well as context. The next two chapters contain our primary technical results, on which the final three chapters containing the principal results of this work are based.

Chapter 1 contains the requisite material on 3-manifold topology that will be assumed throughout. Everything contained therein is standard, and this chapter serves to establish the conventions relied on in the rest of the work.

Chapter 2 reviews Khovanov's construction of a homology theory categorifying the Jones polynomial. We make use of a non-standard normalization natural to our setting, as well as introduce the σ-normalized Khovanov homology. This normalization seems to be interesting, and arises naturally from the work of Manolescu and Ozsváth. We also prove some extensions of this work, obtaining new versions of the skein exact sequence.

Chapter 3 gives a brief outline of Heegaard-Floer homology. It is very difficult to give a complete treatment of this area of intense activity, and we choose to focus on aspects relating to L-spaces and two-fold branched covers. In particular, we give a characterization of Seifert fibered L-spaces which appears to be new.

Chapter 4 develops the necessary material to prove the width bound for branch sets of manifolds with finite fundamental group. In so doing, we prove a surgery result for quasi-alternating knots that seems very natural. In particular, this strengthens the relationship between this class of links (as branch sets) and certain well known L-spaces.

Chapter 5 proves a form of stability for the Khovanov homology of branch sets for integer surgeries on a strongly invertible knot that is analogous to the stable behaviour of Heegaard-Floer homology for large surgeries. This is an essential step in making width a computable surgery obstruction. With this stability lemma in hand, we prove upper and lower bounds for width and establish genericity conditions for which these bounds depend only on the integer fillings.

Chapter 6 states the surgery obstructions derived from Khovanov homology, and gives a range of examples illustrating the application of these obstructions. Notably, we compare our obstructions to some of those provided by the Alexander polynomial (as a result of Heegaard-Floer homology).

Chapter 7 gives some Seifert fibered examples of manifolds that two-fold branch cover the three-sphere in two distinct ways, with branch sets distinguished by the rank of the reduced Khovanov homology. This shows that Khovanov homology is not an invariant of the two-fold branched cover.

Chapter 8 gives various results pertaining to the characterization of the trivial knot. In particular, we establish a large class of knots (containing unknotting number one knots) within which it may be demonstrated that Khovanov homology detects the trivial knot. This is the main ingredient for establishing invariants that detect the trivial knot combining satellites and Khovanov homology. In a similar vein, we give a characterization of the trivial knot, among strongly invertible knots, from Khovanov homology.

The **conclusion** contains some open questions for continued research, and we have included an **appendix** giving an example of our obstructions applied to surgery on a knot in the Poincaré homology sphere.

Conventions and Calculations

Knots with 10 or fewer crossings were tabulated by Bailey and Rolfsen (Rolfsen, 1976, Appendix C), and this work introduced a notation that has now become standard. For knots with 16 or fewer crossings, tabulations are due to Hoste and Thistlethwaite, available via `Knotscape` (Hoste and Thistlethwaite, 1999), with a slightly different notation. As has become standard (see The Knot Atlas (Bar-Natan et al., 2004)), we will use Rolfsen's notation for knots with 10 or fewer crossings, and `Knotscape` notation otherwise.

The examples computed during the course of this research relied heavily on computational software by Shumakovitch (`KhoHo`) (Shumakovitch, 2004a) and Bar-Natan and Greene (`JavaKh`) (Bar-Natan and Green, 2006). The former was an improvement on Bar-Natan's pioneering software, and is an extremely useful tool. However, the speed improvements of `JavaKh` are enough to make the obstructions given in this work practically calculable.

In general, computations given in this thesis were obtained using `JavaKh`.

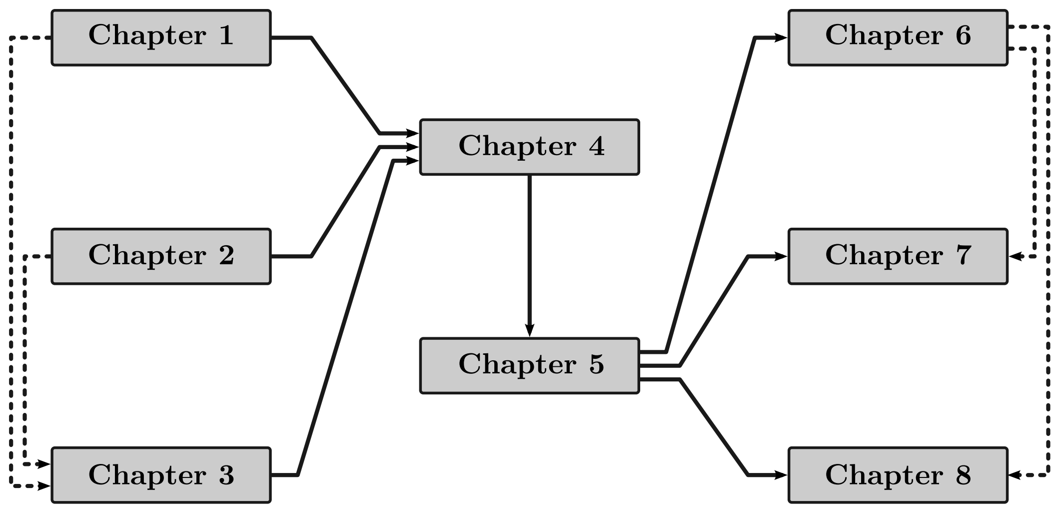


Figure 0.1 Logical dependance: Chapters 1 to 3 give background and are essentially independent, Chapters 4 and 5 establish technical results on which Chapters 6 to 8 (comprising the main results of this thesis) are based. The solid arrows give the dependance of the central material, while the dashed arrows indicate dependance of other results and remarks.

CHAPTER I

DEHN SURGERY ON 3-MANIFOLDS

We begin by briefly outlining the material that will be needed in the sequel pertaining to 3-manifolds and Dehn surgery. All of this material is well-known, and a standard reference is Rolfsen (Rolfsen, 1976). Much of what we will require can be found in the survey paper by Boyer (Boyer, 2002). We endeavour to give accurate references throughout for the results quoted, however for the appropriate historical context we point the reader to Gordon's article on the matter (Gordon, 1999).

1.1 Slopes and fillings

Let M be a compact, connected, orientable 3-manifold with torus boundary.

Definition 1.1. *A slope in ∂M is an element $\alpha \in H_1(\partial M;\mathbb{Z})/\pm 1$, representing the isotopy class of a simple closed curve in ∂M.*

Since $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$, the slopes in ∂M may be parameterized by reduced rational numbers $\{\frac{p}{q}\} \in \mathbb{Q}\cup\{\frac{1}{0}\}$ once a basis (α,β) for $H_1(\partial M;\mathbb{Z})$ has been fixed. That is, any slope may be written in the form $p\alpha+q\beta$ for relatively prime integers p and q, so that the slope α is represented by $\frac{1}{0}$. There is some redundancy in this description that may be taken care of by fixing the convention $q \geq 0$, say. Notice that, as a basis for $H_1(\partial M;\mathbb{Z})$, we have that α and β may be isotoped to intersect transversally in a single point. More generally, it will be useful to measure the distance between any two slopes as follows.

Definition 1.2. *The distance between two slopes* $\alpha, \beta \in H_1(M;\mathbb{Z})/\pm 1$ *is given by their geometric intersection number, denoted* $\Delta(\alpha, \beta)$*.*

As a result, notice that $\Delta(\alpha, \beta) = |\alpha \cdot \beta|$ for any $\alpha, \beta \in H_1(M;\mathbb{Z})/\pm 1$.

Any slope determines a homeomorphism $f_\alpha\colon\, S^1 \times S^1 \to \partial M$, up to isotopy, by specifying $f_\alpha(\mu) = \alpha$ where $\mu = S^1 \times \{\text{point}\}$.

Definition 1.3. *Let* $\mu = \partial D^2 \times \{\text{point}\}$ *in the boundary of a solid torus* $D^2 \times S^1$*. For a given slope* α *on* ∂M *we define the Dehn filling of* M *to be the closed manifold*

$$M(\alpha) = M \cup_{f_\alpha} D^2 \times S^1$$

where the identification of the boundaries is specified by the homeomorphism f_α*.*

1.2 Surgery on knots

Examples of manifolds with torus boundary are given by complements of knots in S^3, and this is where the notion of Dehn filling originates (Dehn, 1910). That is, $M = S^3 \smallsetminus \nu(K)$ where $\nu(K) = D^2 \times K$ is an open tubular neighbourhood of the knot $K \hookrightarrow S^3$. A Dehn filling on such an M in referred to as a Dehn surgery, or simply surgery on the knot K (Rolfsen, 1976, Chapter 9).

In this setting there is a preferred basis for surgery provided by a pair of canonical slopes. First, the knot meridian $\mu = \partial D^2 \times \{\text{point}\}$, and second, the longitude of the knot λ resulting from the fact that K bounds an oriented surface (a Seifert surface) in S^3. That is, any knot $K \hookrightarrow S^3$ comes equipped with a preferred framing given by the intersection of a Seifert surface for K with the boundary ∂M.[1] We may choose orientations on μ and λ so that $\mu \cdot \lambda = 1$, and this convention will be assumed throughout.

Now if α is a slope in ∂M, we may write $\alpha = \pm(p\mu + q\lambda)$ for $q \geq 0$. This gives rise

[1] Indeed, this is always the case when considering a knot in an integer homology sphere, or more generally, a null-homologous knot in any 3-manifold.

to the notation $M(\alpha) = S^3_{p/q}(K)$ for the surgery. Invoking the convention $\frac{1}{0} = \infty$, the trivial surgery $S^3_{1/0}(K) \cong S^3$ is sometimes called the infinity surgery. Pertaining to orientations however, we note that

$$S^3_{p/q}(K) \cong -S^3_{-p/q}(K^\star)$$

where $K^\star$ denotes the mirror image of K, and $-M$ denotes M with opposite orientation. As a result, we may always work with positive surgery coefficients, at the expense of taking mirror images.

By nature of this construction, we have that

$$\pi_1(S^3_{p/q}(K)) \cong \pi_1(M(\alpha)) \cong \pi_1(M) \Big/ \langle\langle \alpha \rangle\rangle,$$

where $\langle\langle \alpha \rangle\rangle$ denotes the normalizer of $\langle \alpha \rangle \subset \pi_1(M)$. And, since $H_1(M;\mathbb{Z}) \cong \mathbb{Z} \cong \langle \mu \rangle$ by Alexander duality,

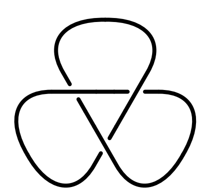
$$H_1(S^3_{p/q}(K);\mathbb{Z}) \cong H_1(M(\alpha);\mathbb{Z}) \cong H_1(M;\mathbb{Z})/\alpha \cong \mathbb{Z}/p\mathbb{Z}.$$

Notice in particular that

$$\left|H_1(S^3_{p/q}(K);\mathbb{Z})\right| = \left|H_1(M(\alpha);\mathbb{Z})\right| = \Delta(\alpha, \lambda)$$

(see, more generally, Lemma 1.5 below).

Example. As a first example, when K is the right-hand trefoil, $S^3_{+1}(K)$ is the Poincaré homology sphere (Poincaré, 1904). Indeed, this is Dehn's original construction of this particular integer homology three-sphere (Dehn, 1910). See (Rolfsen, 1976, Chapter 10) for a detailed account of the equivalence between the constructions of Dehn and Poincaré, and (Kirby and Scharlemann, 1979) for an account of various constructions of this famous 3-manifold.



1.3 The rational longitude

Suppose that $H_1(M;\mathbb{Q}) \cong \mathbb{Q}$, as is the case, for example, when considering the complement of a knot in a rational homology sphere. Such manifolds M will be referred to as *knot manifolds*. Unless stated otherwise, we will generally make the additional assumption that a knot manifold M is irreducible. However, this is not an essential hypothesis in the following discussion, or in the proof of Lemma 1.5 below.

Let $i\colon \partial M \hookrightarrow M$ be the inclusion map, inducing a homomorphism

$$i_*\colon H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q}).$$

Omitting the coefficients for brevity, consider the long exact sequence

$$\cdots \longrightarrow H_2(M) \longrightarrow H_2(M,\partial M) \longrightarrow H_1(\partial M) \xrightarrow{i_*} H_1(M) \longrightarrow H_1(M,\partial M) \longrightarrow \cdots$$

Since ∂M is connected, the inclusion i induces an isomorphism $H_0(\partial M) \cong H_0(M)$. Similarly, since we are working over a field, applying duality $H_3(M,\partial M) \cong H^0(M) \cong H_3(M)$ results in an isomorphism $H_3(M,\partial M) \cong H_2(\partial M)$. Therefore, the sequence simplifies to yield

$$0 \longrightarrow H_2(M) \longrightarrow H_2(M,\partial M) \longrightarrow H_1(\partial M) \xrightarrow{i_*} H_1(M) \longrightarrow H_1(M,\partial M) \longrightarrow 0$$

Since we are working over a field, by duality we have

$$H_2(M) \cong H^1(M,\partial M) \cong H_1(M,\partial M)$$

and

$$H_2(M,\partial M) \cong H^1(M) \cong H_1(M)$$

hence

$$0 \longrightarrow H_1(M,\partial M) \longrightarrow H_1(M) \longrightarrow H_1(\partial M) \xrightarrow{i_*} H_1(M) \longrightarrow H_1(M,\partial M) \longrightarrow 0$$

Now we observe that $\mathrm{rk}(i_*) = 1$. Indeed,

$$\begin{aligned}\mathrm{rk}(i_*) &= b_1(\partial M) - \ker(i_*)\\ &= b_1(\partial M) - b_1(M) + b_1(M, \partial M)\end{aligned}$$

and

$$\mathrm{rk}(i_*) = b_1(M) - b_1(M, \partial M)$$

by exactness. As a result, $b_1(\partial M) = 2\,(b_1(M) - b_1(M, \partial M))$. Now since $b_1(\partial M) = 2$, we conclude that $\mathrm{rk}(i_*) = 1$.

Notice that this implies that $i_*\colon H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ carries a free summand of $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ injectively to $H_1(M;\mathbb{Z}) \cong \mathbb{Z}\oplus H$ (for some finite abelian group H). Moreover, as the image of a free summand of $H_2(M, \partial M;\mathbb{Z})$, $\ker(i_*)$ must be generated by $k\lambda_M$, for some primitive class $\lambda_M \in H_1(\partial M;\mathbb{Z})$, and non-zero integer k.

Note that this class is uniquely defined, up to sign, and hence determines a well-defined slope in ∂M. This gives a canonical slope in ∂M for any knot manifold, and in turn motivates the following definition.

Definition 1.4. *For any knot manifold M, the rational longitude λ_M is the unique slope with the property that $i_*(\lambda_M)$ is finite order in $H_1(M;\mathbb{Z})$.*

More geometrically, the rational longitude λ_M is characterized among all slopes by the property that a non-zero, finite number of like-oriented parallel copies of λ_M bounds an essential surface in M.

1.4 A key lemma

As with the canonical longitude for a knot in S^3, the rational longitude controls the first homology of the manifold obtained by Dehn filling.

Lemma 1.5. *For every knot manifold M there is a constant c_M (depending only on*

M) such that

$$|H_1(M(\alpha);\mathbb{Z})| = c_M \Delta(\alpha, \lambda_M).$$

Proof. Orient λ_M and fix a curve μ dual to λ_M so that $\mu \cdot \lambda_M = 1$. This provides a choice of basis (μ, λ_M) for the group $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Under the homomorphism induced by inclusion we have $i_*(\mu) = (\ell, u)$ and $i_*(\lambda_M) = (0, h)$ as elements of $H_1(M;\mathbb{Z}) \cong \mathbb{Z} \oplus H$. Note that for any other choice of class μ' such that $\mu' \cdot \lambda_M = 1$ we have $\mu' = \mu + n\lambda_M$ so that $i_*(\mu') = (\ell, u + nh)$.

We claim that $|\ell| = \operatorname{ord}_H i_*(\lambda_M)$.

Let ζ generate a free summand of $H_1(M;\mathbb{Z})$ so that (the free part of) the image of μ is $\ell\zeta$ where $i_*(\mu) = (\ell, u) \in \mathbb{Z} \oplus H$, and let η generate the free part of $H_2(M, \partial M;\mathbb{Z})$. Then $\eta \cdot \zeta = \pm 1$ under the intersection pairing $H_2(M, \partial M;\mathbb{Z}) \otimes H_1(M;\mathbb{Z}) \to \mathbb{Z}$.

Now suppose $kh = 0$ where $k = \operatorname{ord}_H i_*(\lambda_M)$ so that the class $k\lambda_M$ bounds a surface in M. The long exact sequence in homology gives

$$\cdots \longrightarrow H_2(M;\mathbb{Z}) \longrightarrow H_2(M, \partial M;\mathbb{Z}) \xrightarrow{\partial} H_1(\partial M;\mathbb{Z}) \xrightarrow{i_*} H_1(M;\mathbb{Z}) \longrightarrow \cdots$$
$$\theta \longmapsto k\lambda_M \longmapsto 0$$

so there is a class $\theta \in H_2(M, \partial M;\mathbb{Z})$ with image $k\lambda_M$. Now we have already observed in defining λ_M that $\operatorname{rk}(i_*) = 1$, and hence $\theta = a\eta$ for some integer $a \neq 0$. Therefore $k\lambda_M = a\partial\eta$, hence $\partial\eta = \frac{k}{a}\lambda_M$. But since $i_*(\frac{k}{a}\lambda_M) = 0$, it must be that $|\frac{k}{a}| = |k|$ so that $|a| = 1$. As a result, $\theta = \pm\eta$. In particular, up to a choice of sign $\partial\eta = k\lambda_M$ as an element of $H_1(\partial M;\mathbb{Z})$. Now

$$|k| = |\mu \cdot k\lambda_M| = |\mu \cdot \partial\eta| = |\ell\zeta \cdot \eta| = |\ell|$$

as claimed.

For a given slope α write $\alpha = a\mu + b\lambda_M$ so that $i_*(\alpha) = (a\ell, au + bh)$. Then

$$H_1(M(\alpha);\mathbb{Z}) \cong H_1(M;\mathbb{Z})/(a\ell, au + bh)$$

has presentation matrix of the form

$$\begin{pmatrix} a\ell & 0 \\ au + bh & Ir \end{pmatrix}$$

where $r = (r_1, \ldots, r_n)$ specifies the finite abelian group $H = \mathbb{Z}/r_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r_n\mathbb{Z}$. Therefore $|H_1(M(\alpha);\mathbb{Z})| = a\ell r_1 \cdots r_n$. Setting

$$c_M = \ell r_1 \cdots r_n = (\mathrm{ord}_H\, i_*(\lambda_M))|H|$$

and noting that $a = \Delta(\alpha, \lambda_M)$ proves the lemma. □

1.5 Heegaard decompositions

A Heegaard decomposition is a decomposition of a 3-manifold along an orientable surface bounding a pair of handlebodies. Such a decomposition exists for any 3-manifold by considering a tubular neighbourhood of the 1-skeleton of a triangulation (Rolfsen, 1976, Chapter 9).

Since our interest, ultimately, will be the role of such decompositions in Heegaard-Floer homology, it is most natural to approach these from the point of view of Morse theory (Milnor, 1963). As such, we follow Ozsváth and Szabó (Ozsváth and Szabó, 2004d) (see also (Ozsváth and Szabó, 2006a, Section 3)).

Fix a Riemannian metric on a closed, connected, orientable 3-manifold Y.

Definition 1.6. *A continuous function on a 3-manifold $f\colon Y \to \mathbb{R}$ is called Morse if all of its critical points are non-degenerate. A Morse function is called self-indexing if for every critical point p we have that $f(p) = \mathrm{index}(p)$. Notice that for a self indexing*

Morse function then, we have $f\colon M \to [0,3]$.

Proposition 1.7. *(Milnor, 1963, Section 6) Every 3-manifold admits a self-indexing Morse function. Further, for a closed, connected, orientable 3-manifold* Y *there is a self-indexing Morse function with a single absolute maximum (critical point of index 3) and a single absolute minimum (critical point of index 0).*

Remark 1.8. *The seminal advance of Morse theory is that the Morse function provides a cellular decomposition of the manifold. As a result, the Morse function may be used to compute the homology of the manifold. With this observation in hand (and the material of (Milnor, 1963) understood), a cellular decomposition of* M *consisting of a single 0-cell and a single 3-cell corresponds to a Morse function of the desired form.*

Now since the Euler characteristic of a closed 3-manifold is zero, there must be the same number of index 2 critical points as index 1 critical points. Furthermore, the level set $f^{-1}(\frac{3}{2})$ is a surface of genus g (given by the size of either of these two sets). The surface $\Sigma_g = f^{-1}(\frac{3}{2})$ then, gives a combinatorial description of the 3-manifold if we record the intersections of flow lines of $-\nabla(f)$ emanating from the index 2 and 1 critical points. That is, the data $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^g$ and $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^g$ are two g-tuples of mutually non-intersecting, essential, simple, closed curves in Σ_g, specifies a 3-manifold uniquely: the α_i specify the 1-handle attachments and the β_i specify the two-handle attachments.

Definition 1.9. *A Heegaard diagram is a triple* $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *consisting of an orientable surface of genus* g*, and two* g*-tuples of mutually non-intersecting, essential, simple, closed curves in* Σ_g.

Notice that, by the existence of a self-indexing Morse function on any 3-manifold, we have that every manifold admits a Heegaard decomposition (c.f. (Ozsváth and Szabó, 2006a, Lemma 3.7)), and that any Heegaard diagram uniquely determines a 3-manifold. In fact, the Heegaard diagram encodes the homology of the manifold as follows:

$$H_1(M;\mathbb{Z}) \cong \frac{H_1(\Sigma_g;\mathbb{Z})}{[\alpha_1],\ldots,[\alpha_g],[\beta_1],\ldots,[\beta_g]}$$

However, by nature of the construction a given 3-manifold may admit many seemingly distinct Heegaard diagrams. It is a result originally due to Singer (Singer, 1933) that any two such diagrams are related by some finite sequence of the following three moves:

- **isotopy:** any α_i may be replaced by an isotopic α_i'. Similarly for the β_i.
- **handleslides:** any α_i may be replaced by an α_i' with the property that there is a triple $(\alpha_i, \alpha_i', \alpha_j)$, disjoint from the remaining α_k, bounding a pair of pants in Σ_g. Similarly for the β_i.
- **stabilization/destabilization:** the genus of the Heegaard surface may be increased by taking $\Sigma_{g+1} = \Sigma_g \# T$, disjoint from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, replacing $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha} \cup \alpha_{g+1}$ and $\boldsymbol{\beta}$ by $\boldsymbol{\beta} \cup \beta_{g+1}$ where $\alpha_{g+1}, \beta_{g+1} \subset T$ intersect in a single point. In a similar manner, we may reduce the genus of the Heegaard surface.

For example, both diagrams in Figure 1.1 give a description of S^3. The genus one description corresponds to a Morse function with a single critical point of each index.

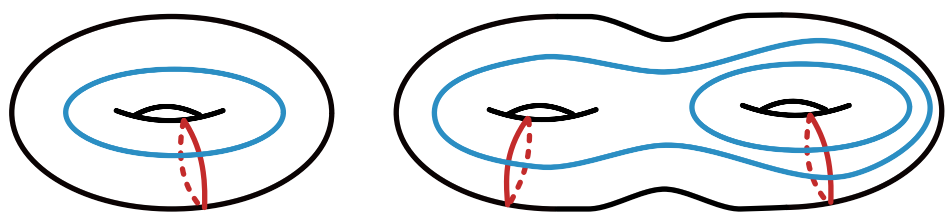


Figure 1.1 The standard genus one decomposition of S^3 (left), and a genus two decomposition resulting from a stabilization, followed by a handleslide (right).

While S^3 is characterized as the only manifold admitting a genus 0 Heegaard decomposition $(S^2, \emptyset, \emptyset)$, lens spaces[2] are characterized as those manifolds admitting Heegaard diagrams of genus 1.

Of course, a neighbourhood of the trivial knot decomposes S^3 into two handlebodies of

[2]Including $S^2 \times S^1$, although we will generally take the viewpoint that this manifold is *not* a lens space.

genus 1. In the interest of fixing our conventions, we conclude this section by comparing the surgery description of these lens spaces, and their Heegaard decompositions.

Consider the standard genus 1 splitting of S^3 given in Figure 1.1. There is a single α curve (in red) and a single β curve (in blue). Considering $S^3 \cong \mathbb{R} \cup \{\infty\}$, let the trivial knot U be the z-axis together with the point at infinity. Then the *solid* torus enclosed by the torus depicted in Figure 1.1 is the the complement of U, and α coincides with the longitude λ (note that α bounds an essential disk). The β curve coincides with the meridian μ of the trivial knot U.

Now the lens space $L(n, 1)$ is given by $S^3_n(U)$, so the corresponding Heegaard diagram for this manifold is

$$(S^1 \times S^1, \alpha = \lambda, \beta = n\mu + \lambda).$$

More generally $S^3_{p/q}(U)$ admits the splitting

$$(S^1 \times S^1, \alpha = \lambda, \beta = p\mu + q\lambda).$$

Remark 1.10. *Our convention that $\mu \cdot \lambda = 1$ in $S^1 \times S^1$ corresponds to the convention that $\mu \cdot \lambda = -1$ when ∂M is oriented as the boundary of M.*

1.6 Cyclic branched covers

Another natural way in which knots and links arise in the study of 3-manifolds is as the fixed point set of a finite group action. That is, given a closed, connected, orientable 3-manifold Y, together with a faithful action by diffeomorphisms $\mathbb{Z}/p\mathbb{Z} \times Y \to Y$ having 1-dimensional fixed point set, the quotient of Y by the action has the structure of an orbifold. In other words, Y may be viewed as a p-fold cyclic branched cover, branched over some link (specified by the image of the fixed point set of the action in the quotient – the orbifold curve).

Such manifolds may be constructed readily, given a knot in S^3. Let $M = S^3 \smallsetminus \nu(K)$, then following (Rolfsen, 1976, Chapter 10), any surjective homomorphism $\pi_1(M) \to \mathbb{Z}/p\mathbb{Z}$

must factor through the abelianization $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$. As a result, the following triangle commutes

$$\begin{array}{ccc} \mu^p & \longmapsto & 0 \\ \pi_1(M) & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ & \searrow \quad \nearrow & \\ & \langle \mu \rangle & \end{array}$$

where μ is the meridian of K, resulting in a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 1$$

Consider the corresponding p-fold cyclic cover $\widetilde{M} \to M$ with $\pi_1(\widetilde{M}) = \Gamma$.

There is a p-fold cyclic branched cover of the disk $D^2 = \{z \in \mathbb{C} \colon |z| \leq 1\}$ by itself specified by $f(z) = z^p$. This extends in an obvious way to a p-fold cyclic branched cover $D^2 \times S^1 \to D^2 \times S^1$, branched over the core $\{0\} \times S^1$ of the solid torus. Here, the core $\{0\} \times S^1$ becomes a singular set of cone index p associated to the p-fold cyclic branched cover (viewed as an orbifold). This extension agrees, on the boundary, with the action of $\mathbb{Z}/p\mathbb{Z}$ restricted to $\partial\widetilde{M}$. With this observation in hand, $Y = \widetilde{M} \cup (D^2 \times S^1)$ gives a p-fold cyclic branched cover of S^3, branched over the knot K. By construction $\pi_1(Y)$ is an index p subgroup of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(S^3, K)$, an object sensitive to the cone index of the singular set K. More precisely, there is a short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1^{\mathrm{orb}}(S^3,K) & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 1 \\ & & \mu & \longmapsto & \mu^p & & & & \end{array}$$

Our interest will be in two-fold branched covers, corresponding to manifolds with involution. To this end we introduce the notation $\mathbf{\Sigma}(S^3, L)$ for the two-fold branched cover of S^3 branched over a link L.[3] In keeping with the discussion above, this notation spec-

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[3]More generally, $\mathbf{\Sigma}(Y, X)$ will denote the two-fold branched cover of Y branched over $X \hookrightarrow Y$, whenever this cover makes sense. Thus, from the present discussion, $D^2 \cong \mathbf{\Sigma}(D^2, \{0\})$ and $D^2 \times S^1 \cong \mathbf{\Sigma}(D^2 \times S^1, \{0\} \times S^1)$.

ifies the singular set L, of cone index 2, when the two-fold branched cover is considered as an orbifold.

Definition 1.11. *A knot manifold is called strongly invertible if there is an involution $f\colon M \to M$ with 1-dimensional fixed point set intersecting the boundary torus transversally in exactly 4 points. A knot is called strongly invertible if its complement is strongly invertible.*

The solid torus is strongly invertible, and by an observation attributed to Montesinos,[4] a strong inversion on M extends to an involution on $M(\alpha)$, for any slope α in ∂M (Montesinos, 1975). This gives a useful relationship between Dehn fillings and two-fold branched covers (c.f. Chapter 4).

Proposition 1.12. *(Montesinos, 1975) For any strongly invertible knot manifold M and slope α in ∂M, the result of Dehn filling gives rise to a two-fold branched cover $M(\alpha) \cong \mathbf{\Sigma}(Y, L)$, for some link $L \hookrightarrow Y$, where Y is the quotient of $M(\alpha)$ by a unique extension of the strong inversion.*

Certain classes of 3-manifolds have a particularly strong correlation with possible branch sets as two-fold branched covers of S^3. For example, the Smith conjecture (resolved in the $\mathbb{Z}/2\mathbb{Z}$ setting by Waldhausen (Waldhausen, 1969)) states that $S^3 \cong \mathbf{\Sigma}(S^3, L)$ if and only if L is the trivial knot. More generally, we have:

Theorem 1.13. *(Hodgson and Rubinstein, 1985) A two-fold branched cover $\mathbf{\Sigma}(S^3, L)$ is a lens space if and only if L is a non-split two-bridge link.*

In a similar vein, work of Boileau and Otal (Boileau and Otal, 1991) gives the following consequence of the orbifold theorem (Thurston, 1982):

Theorem 1.14. *(Boileau and Otal, 1991, Affirmation 2.5) If a two-fold branched $\mathbf{\Sigma}(S^3, L)$ has finite fundamental group, then the branch set $L \hookrightarrow S^3$ is unique up to isotopy.*

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[4]Often colloquially referred to as *the Montesinos trick.*

In fact, it turns out that every manifold with finite fundamental group arises in this way (see Proposition 1.16 and Remark 1.17).

Example: The Poincaré homology sphere may be viewed as the two-fold branched cover $\mathbf{\Sigma}(S^3, K)$ where K is the knot 10_{124} of Rolfsen's tables (Rolfsen, 1976). This knot turns out to be isotopic to the $(3,5)$-torus knot, as well as isotopic to the $(-2,3,5)$-pretzel knot. Of course, this manifold has finite fundamental group the binary icosahedral group, so this observation is consistent with Theorem 1.14.

1.7 Seifert fibered spaces

We now describe a class of manifolds that will play an important role in this work. These were considered by Seifert (Seifert, 1933) and later by Raymond (Raymond, 1968); more details are given in (Boyer, 2002, Section 5.1) and the references therein. In general, the approach taken here follows (Scott, 1983).

A Seifert fibre structure on a 3-manifold M is a foliation by circles. A particular instance of such a structure is given by circle bundles over a surface. Thus, our first example is provided by the solid torus $D^2 \times S^1$. In fact, this manifold admits infinitely many Seifert fibre structures, as follows. Given a pair of relatively prime integers (p,q) with $p \geq 1$, define

$$V_{p,q} = (D^2 \times I)/\{(x,1) = (e^{2\pi i \frac{q}{p}}x, 0)\}.$$

Notice that this describes a foliation by circles induced from the intervals I in the quotient. This is simply a standard solid torus to which a $\frac{p}{q}$-twist has been added. The result is homeomorphic to $D^2 \times S^1$, however the resulting circle foliation is non-standard: the core circle $\{0\} \times S^1 \subset D^2 \times S^1$ is a singular fibre of order p whenever $p > 1$. We take this as the definition of a singular fibre, in general. Work of Epstein shows that every fibered solid torus is fibre-preserving diffeomorphic to one of these standard Seifert fibrations (Epstein, 1972).

Definition 1.15. *A Seifert fibre structure on a 3-manifold M is a foliation by circles* *(called fibres), such that the tubular neighbourhood of any fibre is fibre-preserving diffeomorphic to one of the standard $V_{p,q}$ described above. The index $p \geq 1$ of the fibered solid torus is the index of the fibre; the fibre is singular (or exceptional) whenever $p > 1$, and regular otherwise. A manifold with a fixed Seifert structure will be referred to as a Seifert fibre space.*

The orbit space of a given Seifert fibre space M is an orbifold $\mathcal{B}$, with underlying manifold given by a surface B. The collection of singular fibres of M correspond to a finite collection of cone points in the interior of $\mathcal{B}$, thus we denote $\mathcal{B} = B(p_1, p_2, \ldots, p_n)$. Thus, for a standard fibered solid torus $V_{p,q}$ we have that $\mathcal{B} = D^2(p)$. Notice that the fibres can always be given a coherent orientation locally, but need not admit a global orientation coherent with an orientation on the manifold.

As an example, the Hopf fibration of S^3 demonstrates that this manifold may be viewed as a Seifert fibration. However, the three sphere admits many distinct Seifert structures: one for every torus knot. Consider the genus 1 decomposition of S^3 of Figure 1.1. Then every relatively prime pair (p, q) determines an essential simple closed curve on the torus, or a torus knot when included in S^3. This is a regular fibre in a Seifert fibration with base orbifold $S^2(p, q)$.

The geometry of a closed manifold admitting a Seifert fibration is completely determined by two quantities: the Euler number of the total space and the orbifold characteristic of the base (Scott, 1983, Table 4.1). Thus, for example, we have the following classification:

Proposition 1.16. *(Scott, 1983) A Seifert fibered manifold with finite fundamental group has base orbifold $S^2(p, q, r)$. If $p, q, r > 1$ then these fall into two classes: either $S^2(2, 2, n)$ for any $n > 1$ or $S^2(2, 3, n)$ for $n = 3, 4, 5$.*

Remark 1.17. *Perelman's proof of the geometrization conjecture (Perelman, 2002; Perelman, 2003), carried out in detail by Morgan and Tian (Morgan and Tian, 2007; Morgan and Tian, 2008), implies that the Seifert fibered spaces of Proposition 1.16 entail a complete list of manifolds with finite fundamental group. However, in the case* *when the manifold in question admits a cyclic group action with non-empty fixed point set – as in the case for cyclic branched covers of S^3, in particular – the same may be deduced by avoiding the work of Perelman and applying the orbifold theorem (Thurston, 1982), to obtain a positive resolution to the geometrization conjecture in the presence of a cyclic group action with non-empty fixed point set (see (Boileau and Porti, 2001), and more generally (Boileau et al., 2005) removing the restriction to cyclic group actions). While the present work will make use of this latter fact (see in particular Theorem 4.25 and Remark 4.26), we will endeavour to be explicit when questions pertaining to geometrization arise.*

There is a short exact sequence

$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \pi_1^{\text{orb}}(\mathcal{B}) \longrightarrow 1$$

where $K < \pi_1(M)$ is a cyclic group generated by a regular fibre φ (c.f. (Scott, 1983, Lemma 3.2)). As a result, since $\pi_1(B)$ (and hence $H_1(B;\mathbb{Z})$) is a quotient of $\pi_i^{\text{orb}}(\mathcal{B}) = \pi_1(M)/\langle\langle\varphi\rangle\rangle$, there are strong restrictions on the underlying surface B of the base orbifold $\mathcal{B}$ whenever $H_1(M;\mathbb{Q}) = 0$. Indeed, since surjectivity is preserved under abelianization, the surjection $\pi_1(M) \to \pi_1(B)$ gives a surjection $H_1(M;\mathbb{Z}) \to H_1(B;\mathbb{Z})$. Now if $H_1(M;\mathbb{Z})$ is finite, $H_1(B;\mathbb{Z})$ must be finite as well so that B is either D^2, S^2 or RP^2.

Example: Revisiting our running example of the Poincaré homology sphere, this manifold admits a Seifert fibration with base orbifold $S^2(2,3,5)$. This can be seen from the fact that this manifold is a two-fold branched cover $\mathbf{\Sigma}(S^3,K)$ where K is the $(-2,3,5)$-pretzel knot (Montesinos, 1976) (see also (Boileau and Otal, 1991)). Of course, this knot is isotopic to the $(3,5)$-torus knot, and viewed in this way (see (Seifert, 1933, Page 222)) $\mathbf{\Sigma}(S^3,K)$ is a Brieskorn sphere, that is, the intersection of the unit 5-sphere with the complex surface

$$x^2 + y^3 + z^5 = 0$$

in $\mathbb{C}^3$ (Brieskorn, 1966b; Brieskorn, 1966a) (see (Saveliev, 1999) for a discussion of these notions more closely related to the present work).

Brieskorn spheres provide a nice family of Seifert fibre spaces that branch cover S^3.

Proposition 1.18. *(Milnor, 1975, Lemma 1.1) The Brieskorn sphere resulting from the intersection of S^5 with the complex surface*

$$x^2 + y^p + z^q = 0$$

in $\mathbb{C}^3$ for odd, relatively prime (p, q) is homeomorphic to $\mathbf{\Sigma}(S^3, K)$ where K is the (p,q)-torus knot.

Proposition 1.19. *(Seifert, 1933, Zusatz zu Satz 17) Let K be the (p, q)-torus knot, for p, q odd and relatively prime. Then the two-fold branched cover $\mathbf{\Sigma}(S^3, K)$ admits a unique Seifert fibered structure with base orbifold $S^2(2, p, q)$.*

Remark 1.20. *Though the Seifert structure on this family of manifolds is unique, the involution need not be. In general, if such a manifold has infinite fundamental group, it may be realized as the two-fold branched cover of S^3 in two different ways (Montesinos, 1976).*

We now turn to Dehn surgery on Seifert fibered manifolds. This was studied by Moser (Moser, 1971) in the case of surgery on torus knots in S^3, and subsequently generalized by Heil (Heil, 1974) (see also (Boyer, 2002, Theorem 5.1)).

Theorem 1.21. *(Heil, 1974) Let M be a Seifert fibered knot manifold, with base orbifold $\mathcal{B}$ of the form $B(p_1, p_2, \ldots, p_n)$, where $\partial B = S^1$. Let φ be the slope in ∂M corresponding to a regular fibre of M. Then for any filling $M(\alpha)$, for which $\alpha \neq \varphi$, the Seifert fibration extends and the resulting closed manifold has base orbifold*

$$\mathcal{B} \cup_{\partial B = \partial D^2} D^2 \left(\Delta(\alpha, \varphi)\right).$$

On the other hand,

$$M(\varphi) \cong \left(\#_{i=1}^{n} L(p_i, q_i)\right) \# \left(\#_{j=1}^{m} S^2 \times S^1\right)$$

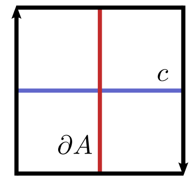
where the i^{th} *fibre is of type* (p_i, q_i) *and* m *is the twice the genus of* B *when* B *is orientable, or the number of* RP^2 *factors otherwise.*

Lens spaces provide a large class of Seifert fibered manifolds; this is now a quick application of the Heil's result. The base orbifold for the lens space $S^3_{p/q}(U) \cong L(p,q)$ is $S^2(p,q) = D^2(p) \cup D^2(q)$ (or if $\frac{p}{q} = 0$ we obtain $S^2 \times S^1$). In other words, for the standard Heegaard decomposition of $L(p,q)$, each solid torus may be fibered with base orbifold $D^2(p)$ and $D^2(q)$ respectively. Note that, due to the nature of this construction and the definition of the $V_{p,q}$, the resulting Seifert structure on a given lens space is highly non-unique.

As we have seen in Lemma 1.5, a Dehn filling is controlled by the rational longitude. However, as in the Theorem 1.21, Seifert fibrations come with a natural choice of slope given by a regular fibre in the boundary. We end this section with a curious collection of examples on which these slopes coincide.

Proposition 1.22. *Let* φ *be a regular fibre in a Seifert fibration on* Y *over* RP^2*, and set* $M = Y \smallsetminus \nu(\varphi)$*. Then the rational longitude coincides with a regular fibre as slopes in* ∂M*.*

Proof. Recall that the rational longitude λ_M is characterized by the following property: some number of parallel copies of λ_M bounds an essential surface in M. Thus, it suffices to show that a regular fibre enjoys this property, and to this end we claim that the class $2\varphi \in H_1(\partial M; \mathbb{Z})$ bounds an essential annulus in M.



To see this, first note that M has base orbifold $\mathcal{B} = B(p_1, p_2, \ldots, p_n)$ where $B = (I \times I)/\{(0,x) = (1, 1-x)\}$ is a Möbius strip. Consider the curve $\{\frac{1}{2}\} \times I$ meeting the boundary in two points. This curve is covered by an annulus $A \hookrightarrow M$ where $\partial A = \varphi \sqcup \varphi \subset \partial M$. Notice that this embedding of A has orientation coherent with the orientation on A, since the fibres above a neighbourhood of the curve $\{\frac{1}{2}\} \times I$ may be coherently oriented.

It remains to see that this surface A is essential, and to this end notice that $c = I \times \{\frac{1}{2}\}$, as a curve in M, meets A transversely in a single point. As a result, since $H^1(M;\mathbb{Z}) \cong H_2(M, \partial M; \mathbb{Z})$ we have the pairing

$$\langle -, - \rangle : \; H_1(M;\mathbb{Z}) \times H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}$$

for which $\langle [c], [A] \rangle \neq 0$. $\square$

Notice that a fibration (without singular fibres) over the Möbius strip may be viewed as the twisted I-bundle over the Klein bottle, $K \widetilde{\times} I$. This I-bundle with base space K is unique (Milnor and Stasheff, 1974), though it admits a second Seifert fibration with base orbifold $D^2(2,2)$. It follows that a Seifert fibration with base orbifold $S^2(2,2,n)$ also admits a Seifert structure over $RP^2(m)$ (for some m). However, such phenomena are the exception, not the rule.

Theorem 1.23. *(Scott, 1983, Theorem 3.9) If Y is a Seifert fibered rational homology sphere with infinite fundamental group, then the Seifert structure is unique.*

1.8 The exceptional surgery problem

To this point, we have discussed essentially combinatorial aspects of 3-manifolds. Our interest however is in questions pertaining to the geometries that arise after Dehn filling. We have seen that, in the case when a manifold admits a Seifert structure, the resulting Dehn fillings are easily understood (as Seifert spaces, c.f. Theorem 1.21), and subsequently the geometry is characterized (Scott, 1983).

In his pioneering work on the geometry and topology of 3-manifolds, Thurston showed that a hyperbolic manifold M with torus boundary admits a finite number of exceptional fillings (Thurston, 1980; Thurston, 1982). That is, those closed manifolds obtained from M by Dehn filling that are non-hyperbolic. Since then, the question of understanding and classifying exceptional surgeries has received considerable attention (see survey papers (Gordon, 1991) and (Boyer, 2002)). What has come to be known as the *exceptional surgery problem* may be stated as follows:

Question 1.24. *Given a hyperbolic 3-manifold M with torus boundary, for which slopes α is $M(\alpha)$ non-hyperbolic?*

Of course, this question may be refined in various ways by asking, for example, when particular geometries arise, or when a particular class of manifolds arises.

Perhaps the simplest non-hyperbolic manifold is a lens space. Restricting to complements of knots in S^3, Moser (Moser, 1971) showed that torus knots always admit lens space surgeries, and went as far as to conjecture that this was the only way to obtain a lens space by surgery on S^3. Subsequently, Bailey and Rolfsen (Bailey and Rolfsen, 1977) constructed an example of a lens space surgery on a non-torus knot (a particular cable of the trefoil),[5] and Fintushel and Stern (Fintushel and Stern, 1980) obtained further examples including hyperbolic knots that admit lens space surgeries.

In a now famous, unpublished note, Berge gives a list of knots in S^3 that admit lens space surgeries (Berge, 1987). These knots are referred to as Berge knots, and it has since been conjectured that this list is complete. That is, if a knot in S^3 admits a lens space surgery then it must be a Berge knot; this has become known as the Berge conjecture.

In this vein, perhaps the most celebrated result pertaining to the exceptional surgery problem is the cyclic surgery theorem due to Culler, Gordon, Luecke and Shalen:

---

[5]Bailey and Rolfsen's article provides an excellent, concise account of Kirby (sometimes referred to as Kirby-Rolfsen) surgery calculus.

Theorem 1.25. *(Culler et al., 1987) Let M be a hyperbolic knot manifold and suppose $M(\alpha)$ and $M(\beta)$ have cyclic fundamental group. Then $\Delta(\alpha, \beta) \leq 1$.*

In particular, this implies that any surgery on S^3 that yields a lens space must be an integer surgery. Further progress towards the Berge conjecture has come, more recently, from applications of Heegaard-Floer homology (Ozsváth and Szabó, 2005b; Rasmussen, 2004b). In fact, there is a active program towards solving the Berge conjecture that has resulted in a completely Heegaard-Floer theoretic version of the conjecture (Baker et al., 2007; Hedden, 2007; Rasmussen, 2007), suggesting that a positive resolution may be possible by way of Heegaard-Floer homology.

Enlarging our class of interest slightly, one might ask instead if a manifold with finite fundamental group can arise as a result of Dehn filling on M. We refer to such a filling as a *finite filling*. This has been treated in depth by Boyer and Zhang, proving the following results analogous to the cyclic surgery theorem, by developing and expanding the machinery and techniques from the proof of Theorem 1.25:

Theorem 1.26. *(Boyer and Zhang, 1996) Let M be a hyperbolic knot manifold. Then if $M(\alpha)$ has finite fundamental group, and $M(\beta)$ has cyclic fundamental group, $\Delta(\alpha, \beta) \leq 2$.*

Theorem 1.27. *(Boyer and Zhang, 2001) Let M be a hyperbolic knot manifold. Then if both $M(\alpha)$ and $M(\beta)$ have finite fundamental group, $\Delta(\alpha, \beta) \leq 3$.*

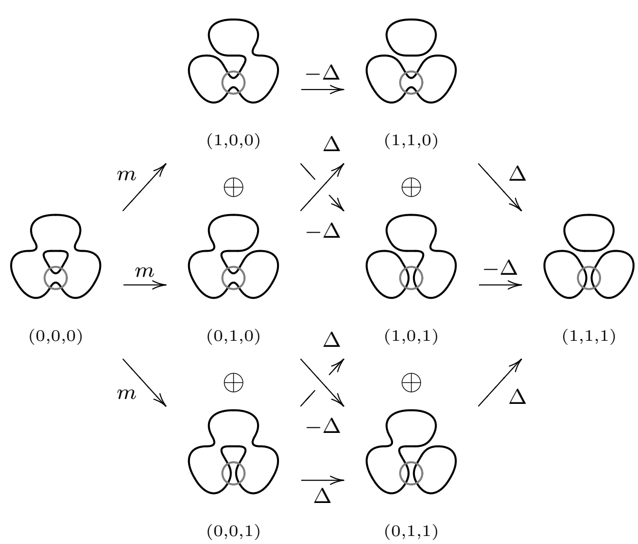
One may proceed in this way, next asking for obstructions to Seifert fibre spaces with base orbifold $S^2(p, q, r)$. Such 3-manifolds are referred to as *small* Seifert fibered spaces. While this is far from a complete treatment of the exceptional surgery question, we pause here to ask the central question of this thesis: can Khovanov homology provide obstructions to exceptional surgeries?

CHAPTER II

KHOVANOV HOMOLOGY

We give a detailed overview of the definition of Khovanov homology (Khovanov, 2000). This has been treated in depth in the literature, and for this reason our introduction is streamlined and tailored to the present purposes. In particular, the proof of invariance will be omitted. We refer the reader to Khovanov's original paper (Khovanov, 2000), as well as (Bar-Natan, 2002; Bar-Natan, 2005). There is also an excellent survey by Rasmussen (Rasmussen, 2005), as well as a very detailed set of notes by Turner from a summer school in Marseille (Turner, 2006).

2.1 Khovanov's construction



The Khovanov complex of an oriented link L is generated by first considering an n-crossing diagram for L together with 2^n states, each of which is a collection of disjoint simple closed curves in the plane. Each state s is obtained from a choice of resolution $\asymp$ (the 0-resolution) or $)($ (the 1-resolution) for every crossing $\times$ (notice that for a crossing the 0- and 1-resolutions exchange roles). As a result, by fixing an order on the crossings, each state s may be represented by an n-tuple with entries in $\{0,1\}$ so that the states are in bijection with the vertices of the n-cube $[0,1]^n$. This is referred to as the cube of resolutions for L. Let $|s|$ denote the *height* of the state s, given by the sum of the entries of the n-tuple associated to s.

Let V be a free, graded $\mathbb{Z}$-module generated by v_- and v_+, where $\deg(v_\pm) = \pm 1$. To each state s we associate $V^{\otimes \ell_s}$ where $\ell_s > 0$ is the number of closed curves in the given state. Set

$$\mathcal{C}^u(L) = \bigoplus_{u=|s|} V^{\otimes \ell_s}[0,|s|].$$

Here, the operator $[\cdot,\cdot]$ shifts the bigrading as follows. Note that we have defined a bigraded group

$$\mathcal{C}(L) = \bigoplus_u \mathcal{C}^u(L) = \bigoplus_{u,q} W^u_q$$

for some finite collection of groups W^u_q where u is the homological (or primary) grading and q denotes the Jones (or secondary) grading. Now the shift operator[1] affects these gradings by

$$(W[i,j])^u_q = W^{u-i}_{q-j}.$$

With this notation in hand, the chain groups of the Khovanov complex are defined as

$$\mathrm{CKh}^u_q(L) = (\mathcal{C}(L)[-n_-, n_+ - 2n_-])^u_q = \mathcal{C}^{u+n_-}_{q-n_++2n_-}(L)$$

where $n_+ = n_+(L)$ is the number of positive crossings in L and $n_- = n_-(L)$ is the number of negative crossings in L.

The differentials $\partial^u\colon \mathrm{CKh}^u(L) \to \mathrm{CKh}^{u+1}(L)$ come from a signed sum over the collection of edges in the cube of resolutions moving from height u to height $u+1$. The

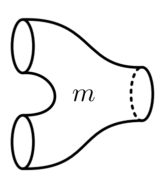
[1]It can be easily verified that this operation corresponds to multiplication in the Poincaré polynomial recording the graded dimensions of these groups. Thus, $V^{\otimes n}$ has Poincaré polynomial $(q^{-1}+q)^n$, where the monomial mq^r denotes that the dimension of the group in q-grading r is m.

operations on each edge correspond to multiplication and comultiplication in a particular Frobenius algebra defined over V.

Notice that each edge in the cube of resolutions connects a pair of states s and s' that differ in precisely one entry. That is, if $|s'| = |s| + 1$ then as elements of $\{0,1\}^n$ these states are of the form $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ldots, \epsilon_n)$ where $\epsilon_k = 0$ for the state s and $\epsilon_k = 1$ for the state s' (the remaining ϵ_i are identical). Geometrically, this corresponds to the local change $\asymp \to)($ (or $)(\to \asymp$), leaving the rest of the state unaltered. Therefore, each edge corresponds to either *merging* two circles of the state s into one to obtain s', or *splitting* a single circle of s in two to obtain s'.

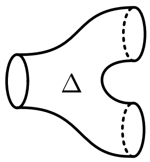
Such operations correspond to simple operations in a cobordism category C with objects given by collections of circles (the states) and arrows given by surfaces. This is a monoidal category $(C, \sqcup, \emptyset)$, and defining a compatible Frobenius algebra amounts to choosing a monoidal functor (i.e. a functor respecting the monoidal structures) to the monoidal category of $\mathbb{Z}$-modules, $(\mathrm{Mod}_{\mathbb{Z}}, \otimes, \mathbb{Z})$. Such a functor is called a TQFT: a topological quantum field theory (Kock, 2004). More precisely, there is an equivalence between isomorphism classes of finite dimensional commutative Frobenius algebras, and isomorphism classes of TQFTs.

We now make the desired Frobenius algebra precise. To each edge of the cube of resolutions we assign the multiplication



$$
\begin{aligned}
m\colon\ V \otimes V &\longrightarrow V \\
v_- \otimes v_- &\longmapsto 0 \\
v_+ \otimes v_- &\longmapsto v_- \\
v_- \otimes v_+ &\longmapsto v_- \\
v_+ \otimes v_+ &\longmapsto v_+
\end{aligned}
$$

whenever the edge $s \to s'$ merges two circles, and we assign the comultiplication



$$\begin{aligned} \Delta \colon V &\longrightarrow V \otimes V \\ v_- &\longmapsto v_- \otimes v_- \\ v_+ &\longmapsto v_- \otimes v_+ + v_+ \otimes v_- \end{aligned}$$

whenever the edge $s \to s'$ splits a single circle.[2] Notice that v_+ is the unit for multiplication, and that each of m and Δ lower degree (the secondary grading) by 1. However, as operations in the cube of resolutions these are grading preserving in q since we have compensated in the definition if $\mathcal{C}(L)$ by shifting in the height $[0, |s|]$.

As a result, viewed as a commutative diagram in the cobordism category C, the cube of resolutions has the property that every 2-dimensional face commutes. To obtain a chain complex then, it suffices to fix a sign convention on the edges so that every 2-dimensional face anti-commutes. In fact, any consistent choice will do, and one such choice is obtained by

$$\text{sign} = \begin{cases} + & \text{if } \sum_{i=1}^{k-1} \epsilon_i \equiv 0 \pmod 2 \\ - & \text{if } \sum_{i=1}^{k-1} \epsilon_i \equiv 1 \pmod 2 \end{cases}$$

where $s = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ldots, \epsilon_n)$ and ϵ_k is the entry changing from 0 to 1 as before. Let ∂_i^u be the operation (with appropriate sign) on the i^{th} edge moving from height u to height $u+1$. Then the differential is defined as

$$\partial^u = \sum \partial_i^u$$

by summing over all edges at the prescribed height. By construction, $(\mathrm{CKh}^u(L), \partial^u)$ forms a chain complex.

---

[2] In fact, the Frobenius algebra (and in particular, the comultiplication) is determined by the multiplication and a counit $\iota : V \to \mathbb{Z}$ defined by $\iota(v_+) = 0$ and $\iota(v_-) = 1$. Though we will not make use of this part of the structure, it is a good check to verify that under this Frobenius algebra the torus evaluates to the map $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$, by observing that v_+ is the unit for multiplication.

9				$\mathbb{Z}$
7				$\mathbb{F}$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			
	0	1	2	3

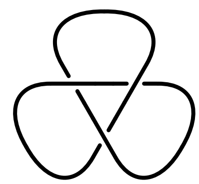


Figure 2.1 The Khovanov homology of the trefoil. The homological grading (u) is read horizontally, and the secondary grading (q) is read vertically. $\mathbb{F}$ denotes the cyclic group $\mathbb{Z}/2\mathbb{Z}$.

Definition 2.1. *The Khovanov homology* $\mathrm{Kh}(L)$ *is given by the homology of the complex* $(\mathrm{CKh}^u(L), \partial^u)$.

Notice that, as defined, $\mathrm{Kh}(L)$ is a bigraded cohomology theory. However, we will continue to refer to Khovanov homology, as has become common in the literature.

Theorem 2.2. *(Khovanov, 2000)* $\mathrm{Kh}(L)$ *is an invariant of the link L, with the property that*

$$\widehat{V}_L(a) = \sum\nolimits_{u,q} (-1)^u a^q \dim(\mathrm{Kh}^u_q(L) \otimes \mathbb{Q})$$

where $\widehat{V}_L(a)$ *is the unnormalized Jones polynomial, with* $\widehat{V}_U(a) = a^{-1} + a$ *for the trivial knot U.*

The proof of the first part of the theorem amounts to showing that the groups $\mathrm{Kh}(L)$ do not depend on the choices made in the construction of $(\mathrm{CKh}^u(L), \partial^u)$, notably, the choice of ordering on the crossings, the sign conventions, and the diagram for the link. In particular, invariance under the three Reidemeister moves[3] must be verified, and this is done in (Khovanov, 2000). A quick proof (working over $\mathbb{Q}$) is given in (Bar-Natan, 2002), and a geometric proof of invariance (of a more general invariant) is given in (Bar-Natan, 2005). A sketch of the proof that blends the two approaches can be found

[3]There are 3 *unoriented* Reidemeister moves; more when equivalence of oriented diagrams is considered.

in (Turner, 2006). The second part of the theorem, pertaining to the Jones polynomial (Jones, 1985), is immediate from the definition of $\mathrm{CKh}(L)$: the Kauffman bracket may be easily recovered from the construction of the cube of resolutions as a graded Euler characteristic of $\mathcal{C}(L)$.

While an *absolute* $\mathbb{Z} \oplus \mathbb{Z}$-grading is a function from homogeneous elements of the homology to $\mathbb{Z} \oplus \mathbb{Z}$, a *relative* $\mathbb{Z} \oplus \mathbb{Z}$-grading is a similar function taking values in the affine space over $\mathbb{Z} \oplus \mathbb{Z}$. That is, only the difference in grading between homogeneous elements is well defined.

As an absolutely graded group, $\mathrm{Kh}(L)$ is concentrated in odd q-gradings whenever L has an odd number of components, and even q-gradings otherwise. As a result, notice that the homology $H_*(\mathcal{C}(L))$ is an invariant of the link as a relatively $\mathbb{Z} \oplus 2\mathbb{Z}$-graded group. This is an invariant of the unoriented link L that has the Kauffman bracket of L as graded Euler characteristic. The Kauffman bracket is an invariant of the link, up to multiplication by some monomial a^k. Thus, the fixed overall shift in $\mathrm{Kh}(L)$ corresponds to adjusting the Kauffman bracket by the writhe to obtain the Jones polynomial (Kauffman, 1987).

Remark 2.3. *Viewed as a relatively $\mathbb{Z} \oplus 2\mathbb{Z}$-graded group, $\mathrm{Kh}(L)$ is still a useful invariant. In particular, it is an invariant of unoriented links.*

Another interesting, basic property of Khovanov homology is that the homology of the mirror $L^\star$ of a link L gives the dual of $Kh(L)$ (Khovanov, 2000, Section 7.3) (see also (Ozsváth and Szabó, 2005c)).

2.2 The skein exact sequence

One of the fundamental tools in Khovanov homology is the skein exact sequence: this is a long exact sequence that plays the role of the skein relation in the Kauffman bracket definition of the Jones polynomial. This exact sequence is implicit in Khovanov's original work (Khovanov, 2000), but appears in the form given here in (Rasmussen, 2005).

Given a link $L(\times)$ with a distinguished positive crossing, fixing an order on the crossings so that this distinguished crossing occurs last, there is a subcomplex

$$\mathcal{C}\left(L()\,()\right)[1,1] \subset \mathcal{C}\left(L(\times)\right)$$

giving rise to a short exact sequence

$$0 \longrightarrow \mathcal{C}\left(L()\,()\right)[1,1] \longrightarrow \mathcal{C}\left(L(\times)\right) \longrightarrow \mathcal{C}\left(L(\asymp)\right) \longrightarrow 0.$$

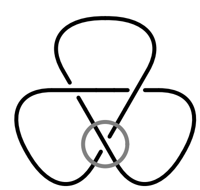
Since $L(\asymp)$ inherits the orientation of $L(\times)$, we set $c = n_-\left(L()\,()\right) - n_-\left(L(\times)\right)$ for some choice of orientation on the affected strands of $L()\,()$ to obtain

$$0 \longrightarrow \mathrm{CKh}^{u-c-1}_{q-3c-2}\left(L()\,()\right) \longrightarrow \mathrm{CKh}^{u}_{q}\left(L(\times)\right) \longrightarrow \mathrm{CKh}^{u}_{q-1}\left(L(\asymp)\right) \longrightarrow 0.$$

This short exact sequence gives rise to a long exact sequence

$$\longrightarrow \mathrm{Kh}^{u-c-1}_{q-3c-2}\left(L()\,()\right) \longrightarrow \mathrm{Kh}^{u}_{q}\left(L(\times)\right) \longrightarrow \mathrm{Kh}^{u}_{q-1}\left(L(\asymp)\right) \xrightarrow{\partial_*} \mathrm{Kh}^{u-c}_{q-3c-2}\left(L()\,()\right) \longrightarrow$$

Here, ∂_* is the map induced on homology from (the component of) the differential $\partial : \mathrm{CKh}^{u}_{q-1}\left(L(\asymp)\right) \to \mathrm{CKh}^{u-c}_{q-3c-2}\left(L()\,()\right)$ in $\mathrm{CKh}^{u}_{q}\left(L(\times)\right)$. This connecting homomorphism raises homological degree by one, and preserves the secondary grading.



For example, in the complex for the right-hand trefoil (given in the previous section) we have circled the distinguished positive crossing (it is shown on the right). The subcomplex is given by states of the form $(\star,\star,1)$, and ∂_* is induced by morphisms that take states of the form $(\star,\star,0)$ to $(\star,\star,1)$. It is an instructive exercise to calculate that $\mathrm{Kh}^7_3(K) \cong \mathbb{Z}/2\mathbb{Z}$ for the right hand trefoil K. This results from the fact that the only non-trivial morphism in this long exact sequence arises for $\mathrm{Kh}^6_2(H)[0,1] \to \mathrm{Kh}^0_{-1}(U)[3,8]$, where H is the Hopf link and U is the trivial knot, and turns out to be $\times 2 : \mathbb{Z} \to \mathbb{Z}$ (see (Turner, 2006) for more details on this example). Torsion in Khovanov homology is somewhat mysterious, though it is conjectured that the torsion alone is enough to detect that a knot is non-trivial (Shumakovitch, 2004b).

Similarly, for a link $L(\diagdown\!\!\!\!\diagup)$ with a distinguished negative crossing there is a long exact sequence

$$\longrightarrow \mathrm{Kh}^{u}_{q+1}\left(L(\asymp)\right) \longrightarrow \mathrm{Kh}^{u}_{q}\left(L(\diagdown\!\!\!\!\diagup)\right) \longrightarrow \mathrm{Kh}^{u-c}_{q-3c-1}\left(L()\,()\right) \xrightarrow{\partial_*} \mathrm{Kh}^{u+1}_{q+1}\left(L(\asymp)\right) \longrightarrow$$

2.3 Reduced Khovanov homology

Given a link L, there is a reduction of the chain complex defined for $L_\bullet$: the link with a choice of marked arc in a diagram for L (Khovanov, 2003). This depends, in general, on a choice of marked component, but gives a well defined invariant for knots. We give two equivalent definitions.

The multiplication m gives rise to an action $V \otimes \mathrm{CKh}(L_\bullet) \to \mathrm{CKh}(L_\bullet)$ by stipulating that a closed component introduced near the marked point merges at that point under the obvious cobordism. We note that the unit for multiplication acts trivially, and that the associativity of V ensures that the action is well defined. As a result, $\mathrm{CKh}(L_\bullet)$ is a complex of V-modules.

Definition 2.4. *The reduced Khovanov homology of $L_\bullet$, denoted $\widetilde{\mathrm{Kh}}(L_\bullet)$, is given by the homology of the complex $\widetilde{\mathrm{CKh}}(L_\bullet) = \mathrm{CKh}(L_\bullet) \otimes_V V\big/(v_- \cdot V)$.*

The reduced Khovanov homology is an invariant of $L_\bullet$, depending in general on the marked component. As a result, we get a well defined invariant when restricting attention to knots (i.e. single component links).

There is also a natural way to view this reduction in terms of subcomplexes. The marking on $L_\bullet$ descends to a marking of states $s_\bullet$. Since v_+ is the unit for multiplication, we may form a subcomplex $\mathcal{C}(L_\bullet) \subset \mathcal{C}(L)$ as follows:

$$\mathcal{C}^u(L_\bullet) = \sum_{u=|s|} v_- \otimes V^{\otimes(\ell_s - 1)}$$

where the marked circle in the state is always endowed with the element $v_- \in V$. That $\mathcal{C}(L_\bullet)$ is a subcomplex is immediate from the definition of m and Δ in the associated Frobenius algebra. As a result, we may define $\mathrm{CKh}_\bullet(L_\bullet) = \bigoplus_u \mathcal{C}^u(L_\bullet)$ to obtain the short exact sequence

$$0 \longrightarrow \mathrm{CKh}_\bullet(L_\bullet) \longrightarrow \mathrm{CKh}(L) \longrightarrow \widetilde{\mathrm{CKh}}(L_\bullet) \longrightarrow 0$$

where $\widetilde{\mathrm{CKh}}(L_\bullet) \cong \mathrm{CKh}(L)/\,\mathrm{CKh}_\bullet(L_\bullet)$ is taken as the definition of the reduced Khovanov complex. Indeed, this is precisely the tensor product (over V) with the one dimensional representation $V\big/(v_- \cdot V)$ given previously. As before, the homology of this complex is denoted $\widetilde{\mathrm{Kh}}(L_\bullet)$.

Theorem 2.5. *(Khovanov, 2003)* $\widetilde{\mathrm{Kh}}(L_\bullet)$ *is an invariant of the marked link* $L_\bullet$ *(and in particular, gives an invariant for knots) with the property that*

$$V_L(t) = \sum\nolimits_{u,q} (-1)^u t^{\frac{q}{2}} \operatorname{rk}(\widetilde{\mathrm{Kh}}(L) \otimes \mathbb{Q})$$

where $V_L(t)$ *is the standard Jones polynomial with normalization* $V_U(t) = 1$ *for the trivial knot* U.

The short exact sequence for the reduced complex gives rise to a long exact sequence of the form

$$\longrightarrow \widetilde{\mathrm{Kh}}^u(L_\bullet)[0,1] \longrightarrow \mathrm{Kh}^u(L) \longrightarrow \widetilde{\mathrm{Kh}}^u(L_\bullet)[0,-1] \longrightarrow \widetilde{\mathrm{Kh}}(L_\bullet)^{u+1}[0,1] \longrightarrow$$

since $\mathrm{CKh}_\bullet(L_\bullet) \cong \widetilde{\mathrm{CKh}}(L_\bullet)[0,2]$ (see (Rasmussen, 2005)). While this sequence does not split in general, work of Shumakovitch implies that the connecting homomorphism is relatively tame.

Theorem 2.6. *(Shumakovitch, 2004b) The connecting homomorphism in the long exact sequence for the reduced complex is congruent to 0 modulo 2.*

Thus, working with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ we have a split long exact sequence.

2.4 Coefficients and further conventions

For our purposes, it is not restrictive to work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. And indeed, the benefits are such that we will fix this choice once and for all. As a result, we immediately have that $\widetilde{\mathrm{Kh}}(L)$ is an invariant of *unmarked* links (i.e. does not depend on the choice of marked component), and

$$\mathrm{Kh}(L) \cong \widetilde{\mathrm{Kh}}(L)[0,-1] \oplus \widetilde{\mathrm{Kh}}(L)[0,1]$$

(Shumakovitch, 2004b).

We now fix some conventions for the remainder of this work. Replacing the secondary grading by $\frac{q}{2}$ (but preserving the notation q for this rescaling), we define $\delta = u - q$. Now we consider $\widetilde{\mathrm{Kh}}(L)$ as a relatively $\mathbb{Z}\oplus\mathbb{Z}$-graded homology theory in gradings δ and q, so that group that until now has been written $\widetilde{\mathrm{Kh}}^i_j(L)$ will from now on be denoted $\widetilde{\mathrm{Kh}}^\delta_q(L)$ where $\delta = i - \frac{j}{2}$ and $q = \frac{j}{2}$.

As a relatively graded group, this homology theory categorifies to the Jones polynomial in the following sense (c.f. Theorem 2.2 and Theorem 2.5).

Theorem 2.7. *Let $u = \delta + q$. Then there is a unique absolute $\mathbb{Z}\oplus\frac{1}{2}\mathbb{Z}$-grading (in (u,q)) on $\widetilde{\mathrm{Kh}}(L)$ with the property that*

$$V_L(t) = \sum\nolimits_{u,q} (-1)^u t^q \operatorname{rk} \widetilde{\mathrm{Kh}}^u_q(L),$$

where $V_L(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ is the Jones polynomial.

Remark 2.8. *We remark that the universal coefficient theorem*

$$\widetilde{\mathrm{Kh}}^u_q(L;\mathbb{F}) \cong \mathrm{Tor}\left(\widetilde{\mathrm{Kh}}^{u+1}_q(L;\mathbb{Z}),\mathbb{F}\right) \oplus \widetilde{\mathrm{Kh}}^u_q(L;\mathbb{Z}) \otimes \mathbb{F}$$

together with the fact that

$$\mathrm{Tor}(\mathbb{Z}/2n\mathbb{Z},\mathbb{F}) = \mathbb{F} = \mathbb{Z}/2n\mathbb{Z} \otimes \mathbb{F}$$

ensures that $\operatorname{rk}\widetilde{\mathrm{Kh}}(L;\mathbb{F}) = \operatorname{rk}\widetilde{\mathrm{Kh}}(L;\mathbb{Q}) + 2\ell$*, for some integer* $\ell \geq 0$*, and the extra factors of* $\mathbb{F}$ *cancel in pairs so that the graded Euler characteristic (giving rise to the Jones polynomial) is invariant of the coefficient field.*

Thus, the Jones polynomial arises as an appropriately defined *graded* Euler characteristic of the theory. According to our grading conventions, the usual Euler characteristic

$$\chi(\widetilde{\mathrm{Kh}}(L)) = \textstyle\sum_{\delta}(-1)^{\delta}\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}(L)$$

is obtained by collapsing the q grading. Note that this is only well defined up to sign as δ is a relative integer grading; we fix the convention $\chi \geq 0$.

Proposition 2.9. *With the above notation,* $\chi(\widetilde{\mathrm{Kh}}(L)) = \det(L)$ *(the standard determinant of the link).*

Proof.

$$\begin{aligned}
\chi(\widetilde{\mathrm{Kh}}(L)) &= \Big|\sum_{\delta}(-1)^{\delta}\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}(L)\Big| \\
&= \Big|\sum_{\delta,q}(-1)^{\delta}\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}_{q}(L)\Big| \\
&= \Big|\sum_{u,q}(-1)^{u-q}\operatorname{rk}\widetilde{\mathrm{Kh}}^{u}_{q}(L)\Big| \\
&= \Big|\sum_{u,q}(-1)^{u}(-1)^{-q}\operatorname{rk}\widetilde{\mathrm{Kh}}^{u}_{q}(L)\Big| \\
&= \Big|\sum_{u,q}(-1)^{u}(-1)^{q}\operatorname{rk}\widetilde{\mathrm{Kh}}^{u}_{q}(L)\Big| \\
&= |V_L(-1)| \\
&= \det(L)
\end{aligned}$$

□

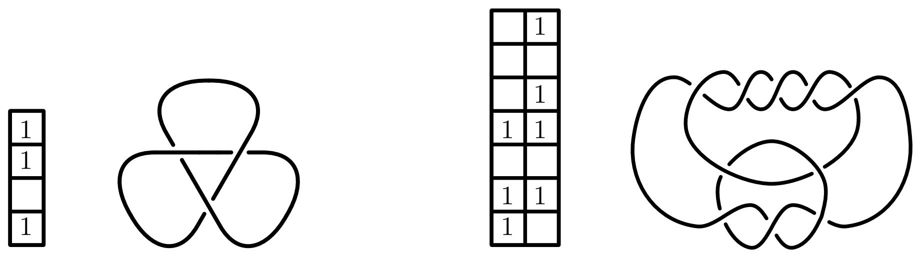


Figure 2.2 The reduced Khovanov homology of the trefoil (left) with $w = 1$, and the knot 10_{124} (right) with $w = 2$. The primary relative grading (δ) is read horizontally, and the secondary relative grading (q) is read vertically. The values at a given bi-grading give the ranks of the abelian group (or $\mathbb{F}$-vector space) at that location; trivial goups are left blank.

Forgetting the q-grading in this way, and collecting the δ-gradings, yields

$$\widetilde{\mathrm{Kh}}(L) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k} = \textstyle\bigoplus_{\delta=1}^{k} \mathbb{F}^{b_\delta}$$

for non-negative integers b_i, where b_1 and b_k are non-zero. As a result, we arrive naturally at the following:

Definition 2.10. *The homological width of L is given by $w(L) = k$, the number of δ-gradings supporting the reduced Khovanov homology. Links for which $w = 1$ are called thin (or homologically thin), while links with $w > 1$ are termed thick (or homologically thick).*

Hence, our grading convention gives homological grading by diagonals of slope 2 from the standard (u, q)-grading related to the Jones polynomial.

Remark 2.11. *The homological width is an interesting quantity. It is an invariant of the link taking values in $\mathbb{N}$ that cannot be recovered from the Jones polynomial. Bar-Natan's calculation of $\mathrm{Kh}(K)$ for knots with up to 11 crossings (Bar-Natan, 2002) first suggested that the quantity is of interest (see also (Khovanov, 2003; Shumakovitch, 2004b)), and his conjecture that $w(L) = 1$ for non-split alternating links was subsequently proved by Lee (Lee, 2005). Of course, these are not the only homologically thin links. In general the quantity $w(L)$ seems mysterious, and worthy of study as a result.*

With this notation in hand,

$$\chi\left(\widetilde{\mathrm{Kh}}(L)\right) = \Big|\sum_{i=\delta}^{k}(-1)^{\delta}b_{\delta}\Big|$$

and

$$\operatorname{rk}\widetilde{\mathrm{Kh}}(L) = \sum_{\delta=1}^{k} b_{\delta},$$

giving rise to a first example of homologically thick links.

Proposition 2.12. *Any link L with $\det(L) = 0$ must have $w(L) > 1$.*

Proof. Since $V_L(t)$ is a non-zero polynomial[4] it follows that $\operatorname{rk}\widetilde{\mathrm{Kh}}(L) > 0$ for all links L, and in particular that there is at least one $b_\delta \neq 0$. But since $\det(L) = \chi\big(\widetilde{\mathrm{Kh}}(L)\big) = 0$, there must be at least two such gradings supporting non-trivial groups. As a result, determinant 0 links are homologically thick. $\square$

2.5 Mapping cones and exact triangles

The skein exact sequence for reduced Khovanov homology – which exists as a result of the observation that $\widetilde{\mathrm{Kh}}(L)$ may be viewed as the homology of a subcomplex of $\mathrm{CKh}(L)$ – carries over directly to our grading conventions (see also (Rasmussen, 2005; Manolescu and Ozsváth, 2007)). For a link $L(\nwarrow\!\!\!\!\nearrow)$ with distinguished positive crossing we have that

$$\longrightarrow \widetilde{\mathrm{Kh}}\left(L()\,()\right)[-\tfrac{1}{2}c, \tfrac{1}{2}(3c+2)] \longrightarrow \widetilde{\mathrm{Kh}}\left(L(\times)\right) \longrightarrow \widetilde{\mathrm{Kh}}\left(L(\asymp)\right)[-\tfrac{1}{2}, \tfrac{1}{2}] \longrightarrow$$

and for a link with distinguished negative crossing $L(\searrow\!\!\!\!\nearrow)$ we have

$$\longrightarrow \widetilde{\mathrm{Kh}}\left(L(\asymp)\right)[\tfrac{1}{2}, -\tfrac{1}{2}] \longrightarrow \widetilde{\mathrm{Kh}}\left(L(\times)\right) \longrightarrow \widetilde{\mathrm{Kh}}\left(L()\,()\right)[-\tfrac{1}{2}(c+1), \tfrac{1}{2}(3c+1)] \longrightarrow$$

Omitting grading shifts for the moment, and simplifying with the notation $\times$ for $L(\times)$,

---

[4] Jones shows that $V_L(1) = 2^{|L|-1}$ (Jones, 1985, Theorem 15), hence $V_L(t) \neq 0$ for any link L.

these exact sequences are often represented by exact triangles of the form

$$\begin{array}{ccc} & \widetilde{\mathrm{Kh}}(\times) & \\ \nearrow & & \searrow \\ \widetilde{\mathrm{Kh}}()\,() & \xleftarrow{[1,0]} & \widetilde{\mathrm{Kh}}(\asymp) \end{array}$$

Since we are working over a field, the homology $\widetilde{\mathrm{Kh}}(L)$ is completely determined by the groups $\widetilde{\mathrm{Kh}}(\asymp)$ and $\widetilde{\mathrm{Kh}}()\,()$, together with the connecting homomorphism. This leads directly to the notion of a mapping cone (see (Weibel, 1994, Chapter 1), for example, or (Ozsváth and Szabó, 2005c, Section 4)), which will be a useful point of view in the sequel. That is

$$\widetilde{\mathrm{CKh}}(\times) \cong \left(\widetilde{\mathrm{CKh}}(\asymp) \oplus \widetilde{\mathrm{CKh}}()\,()\right., D\left.\right)$$

where

$$D = \begin{pmatrix} \partial_0 & 0 \\ \partial & \partial_1 \end{pmatrix}$$

is a differential (since we are working over $\mathbb{F}$) composed of the differential on $\widetilde{\mathrm{CKh}}(\asymp)$ (denoted ∂_0), the differential on $\widetilde{\mathrm{CKh}}()\,()$ (denoted ∂_1), and ∂, the component of the differential inducing the connecting homomorphism. Passing to homology, we have that

$$\widetilde{\mathrm{Kh}}(\times) \cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp) \to \widetilde{\mathrm{Kh}}()\,()\right)$$

where the connecting homomorphism raises homological δ-grading by one.

Replacing the grading shifts (in terms of δ and q), we have

$$\widetilde{\mathrm{Kh}}(\times) \cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}, \tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c, \tfrac{1}{2}(3c+2)]\right)$$
$$\widetilde{\mathrm{Kh}}(\times) \cong H_*\left(\widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}(c+1), \tfrac{1}{2}(3c+1)] \to \widetilde{\mathrm{Kh}}(\asymp)[\tfrac{1}{2}, -\tfrac{1}{2}]\right)$$

The singly δ-graded group will be useful in many instances, and in this setting the mapping cones simplify to yield

$$\begin{aligned}
\widetilde{\mathrm{Kh}}(\slashoverback) &\cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c]\right)\\
\widetilde{\mathrm{Kh}}(\backoverslash) &\cong H_*\left(\widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}(c+1)] \to \widetilde{\mathrm{Kh}}(\asymp)[\tfrac{1}{2}]\right)
\end{aligned}$$

where $[\cdot]$ shifts the δ-grading.

2.6 Normalization and Support

In calculations involving the skein exact sequence absolute gradings are essential. Therefore, we will generally need to fix an orientation, although the final result will not depend on this choice so long as we remain consistent, according to Remark 2.3.

In particular, $w(L)$ depends only on $\widetilde{\mathrm{Kh}}(L)$ as a relatively graded group, however determining this quantity in practice will depend on absolute gradings. For this reason we introduce the notion of support $\mathrm{Supp}(\widetilde{\mathrm{Kh}}(L))$ as an absolutely $\mathbb{Z}$-graded quantity. Thus if

$$\widetilde{\mathrm{Kh}}(\slashoverback) \cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c]\right)$$

and $\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}()\,()[-\tfrac{1}{2}c]\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}]\right)$ then we may write

$$\widetilde{\mathrm{Kh}}(\slashoverback) \cong H_*\left(\begin{array}{ccccccc}
\mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k}\\
 & \searrow & & \searrow & & \searrow & \\
\mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_k}
\end{array}\right)$$

for $b_i \geq 0$, since the connecting homomorphism raises δ-grading by 1.

The following will be a useful absolutely $\mathbb{Z}$-graded object:

Definition 2.13. *The σ-normalized Khovanov homology is an absolutely $\mathbb{Z}$-graded theory defined by $\widetilde{\mathrm{Kh}}_\sigma(L) = \widetilde{\mathrm{Kh}}(L)[-\frac{\sigma(L)}{2}]$ where $\sigma(L)$ denotes the signature of the link L.*

This turns out to be a natural grading to consider, despite the fact that we are interested only in the relative grading, ultimately. Of course, $\widetilde{\mathrm{Kh}}_\sigma(L)$ and $\widetilde{\mathrm{Kh}}(L)$ coincide as relatively $\mathbb{Z}$-graded groups.

2.7 The Manolescu-Ozsváth exact sequence

As a singly graded theory, there is a useful special case in which the skein exact sequence simplifies nicely in terms of the σ-normalization.

Proposition 2.14. *(Manolescu and Ozsváth, 2007, Proposition 5) Let $L = L(\times)$ be a link with some distinguished crossing, and set $L_0 = L(\asymp)$ and $L_1 = L()\,()$. If* $\det(L_0), \det(L_1) > 0$ *and* $\det(L) = \det(L_0) + \det(L_1)$ *then*

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0) \to \widetilde{\mathrm{Kh}}_\sigma(L_1)\right).$$

In the standard notation, this takes the form

$$\widetilde{\mathrm{Kh}}(L)[-\tfrac{\sigma}{2}] = H_*\left(\widetilde{\mathrm{Kh}}(L_0)[-\tfrac{\sigma_0}{2}] \to \widetilde{\mathrm{Kh}}(L_1)[-\tfrac{\sigma_1}{2}]\right).$$

where $\sigma = \sigma(L)$, $\sigma_0 = \sigma(L_0)$ and $\sigma_1 = \sigma(L_1)$ (see (Manolescu and Ozsváth, 2007)).

Notice, in particular, that in this setting the orientation of the resolved crossing does not play a role and the pair of exact sequences have a single expression.

2.8 A digression on the signature of a link

We briefly review the work of Gordon and Litherland, constructing the signature of a link via the Goeritz matrix (Gordon and Litherland, 1978). The conventions we adopt are those of (Manolescu and Ozsváth, 2007), since our interest will be in proving a degenerate form of Proposition 2.14.

The complement of a projection of a link L is divided into regions that may be coloured black and white in an alternating fashion to obtain the checkerboard colouring. Denote the white regions by $R_0, R_1, \ldots, R_n$. By eliminating nugatory crossings,[5] we may assume that every crossing c of the diagram for L is incident to distinct white regions, and assign an incidence number $\mu(c)$ and type by the conventions of Figure 2.3.

Figure 2.3 Incidence numbers and crossing types.

The incidence number of the diagram for L is obtained by taking the sum of incidences over crossings of type II. Setting

$$\mu(L) = \sum_{c \text{ of type II}} \mu(c),$$

the Goeritz matrix of G for the diagram of L is the $n \times n$ symmetric matrix

$$g_{ij} = \begin{cases} -\sum_{c \in R_{ij}} \mu(c) & i \neq j \\ -\sum_{i \neq k} g_{ik} & i = j \end{cases}$$

where $R_{ij} = \overline{R_i} \cap \overline{R_j}$ for $i, j \in \{1, \ldots, n\}$.

Then the signature of the link L is given by

$$\sigma(L) = \text{signature}(G) - \mu(L)$$

and

$$\det(L) = |\det(G)|$$

(Gordon and Litherland, 1978).

[5] This amounts to applying Reidemeister I moves.

2.9 Degenerations

We now prove that Manolescu and Ozsváth's exact sequence degenerates (in a very controlled manner) when one of the pair of determinants vanishes. Once again, a single expression is obtained in each case.

Proposition 2.15. *Using the same conventions as Proposition 2.14, if* $\det(L_0) = 0$ *and* $\det(L) = \det(L_1) \neq 0$ *then*

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(L_1)\right).$$

Similarly, if $\det(L_1) = 0$ *and* $\det(L) = \det(L_0) \neq 0$ *then*

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0) \to \widetilde{\mathrm{Kh}}_\sigma(L_1)[\tfrac{1}{2}]\right).$$

Proof. The proof closely follows the argument in (Manolescu and Ozsváth, 2007) establishing Proposition 2.14, and as such we will adopt the same notation. Throughout, $\sigma = \sigma(L)$, $\sigma_0 = \sigma(L_0)$ and $\sigma_1 = \sigma(L_1)$. There are 2 orientations to consider in each case, hence 4 cases to consider in total.

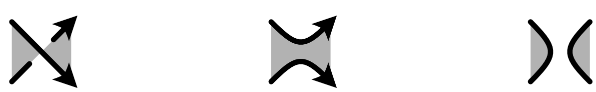


Figure 2.4 Colouring conventions for case 1: L, L_0 (the oriented resolution) and L_1 (the unoriented resolution) at the resolved positive crossing. For case 2 the white and black regions are exchanged to yield the dual colouring.

Case 1: Suppose the distinguished crossing is positive, with $\det(L_0) = 0$, and fix a checkerboard colouring of the diagram for L as in Figure 2.4 so that the distinguished crossing is of type II with incidence $\mu = +1$. Now writing G_1 for the Goeritz matrix of L_1, we have

$$G = \begin{pmatrix} a & v \\ v^T & G_1 \end{pmatrix} \quad \text{and} \quad G_0 = \begin{pmatrix} a-1 & v \\ v^T & G_1 \end{pmatrix}$$

where G and G_0 are the Goeritz matrices of L and L_0 respectively. As in (Manolescu and Ozsváth, 2007), we assume without loss of generality that G_1 is diagonal (with diagonal entries $\alpha_1, \ldots, \alpha_n$) and write the bilinear form associated to G as

$$\left(a - \sum_{i=1}^{n} \frac{v_i^2}{\alpha_i}\right) x_0^2 + \sum_{i=1}^{n} \alpha_i \left(x_i + \frac{v_i}{\alpha_i} x_0\right)^2.$$

Similarly, the bilinear form associated to G_0 may be written as

$$\left(a - 1 - \sum_{i=1}^{n} \frac{v_i^2}{\alpha_i}\right) x_0^2 + \sum_{i=1}^{n} \alpha_i \left(x_i + \frac{v_i}{\alpha_i} x_0\right)^2$$

so that setting

$$\beta = a - \sum_{i=1}^{n} \frac{v_i^2}{\alpha_i}$$

we obtain

$$\det(G) = \beta \det(G_1) \text{ and } \det(G_0) = (\beta - 1) \det(G_1).$$

Now since $0 = \det(L_0) = |\det(G_0)| = |\beta - 1| \det(L_1)$ and $\det(L_1) \neq 0$, we have that $\beta = +1$ and

$$\text{signature}(G) = \text{signature}(G_0) + 1 = \text{signature}(G_1) + 1.$$

Using the Gordon-Litherland formula for the signature we have that

$$\begin{aligned} \sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_0) + 1 - (\mu_0 + 1) \\ &= \sigma_0 \end{aligned}$$

where $\mu = \mu(L)$ and $\mu_0 = \mu(L_0)$, while writing $\mu_1 = \mu(L_1)$ gives

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_1) + 1 - (\mu_1 + c + 1) \\ &= \sigma_1 - c\end{aligned}$$

as in (Manolescu and Ozsváth, 2007), noting that the incidence and type of a crossing determines its sign. Now since

$$\widetilde{\mathrm{Kh}}\,(L) \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{c}{2}]\right)$$

we have $-1 = \sigma - \sigma_0 - 1$ and $-c = \sigma - \sigma_1$ so that

$$\widetilde{\mathrm{Kh}}\,(L)\,[-\tfrac{\sigma}{2}] \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{\sigma_0+1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{\sigma_1}{2}]\right).$$

In terms of the σ-normalization,

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(L_1)\right)$$

as claimed.

Case 2: If once again we consider a positive distinguished crossing, but instead the resolution L_1 has $\det(L_1) = 0$, then fix the dual colouring to that of Figure 2.4 so that the distinguished crossing is of type I with incidence $\mu = -1$. Now letting G, G_0 and G_1 be the Goeritz matrices for L, L_0 and L_1 respectively, we have that

$$G = \begin{pmatrix} a & v \\ v^T & G_0 \end{pmatrix} \quad \text{and} \quad G_1 = \begin{pmatrix} a+1 & v \\ v^T & G_0 \end{pmatrix}$$

Diagonalizing yields

$$\det(G) = \beta \det(G_0) \text{ and } \det(G_1) = (\beta + 1) \cdot \det(G_0)$$

so our hypothesis forces $\beta = -1$, resulting in

$$\text{signature}(G) = \text{signature}(G_0) - 1 = \text{signature}(G_1) - 1.$$

Therefore,

$$\begin{aligned} \sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_0) - 1 - \mu_0 \\ &= \sigma_0 - 1 \end{aligned}$$

while

$$\begin{aligned} \sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_1) - 1 - (\mu_1 + c) \\ &= \sigma_1 - c - 1 \end{aligned}$$

so that $-1 = \sigma - \sigma_0$ and $c = \sigma - \sigma_1 + 1$. Thus

$$\widetilde{\text{Kh}}\,(L) \cong H_*\left(\widetilde{\text{Kh}}\,(L_0)\,[-\tfrac{1}{2}] \to \widetilde{\text{Kh}}\,(L_1)\,[-\tfrac{c}{2}]\right)$$

yields

$$\widetilde{\text{Kh}}\,(L)\,[-\tfrac{\sigma}{2}] \cong H_*\left(\widetilde{\text{Kh}}\,(L_0)\,[-\tfrac{\sigma_0}{2}] \to \widetilde{\text{Kh}}\,(L_1)\,[-\tfrac{\sigma_1 - 1}{2}]\right).$$

In terms of the σ-normalization,

$$\widetilde{\text{Kh}}_\sigma(L) = H_*\left(\widetilde{\text{Kh}}_\sigma(L_0) \to \widetilde{\text{Kh}}_\sigma(L_1)[\tfrac{1}{2}]\right)$$

as claimed.

Case 3: Suppose the distinguished crossing is negative, with $\det(L_1) = 0$; the argument varies only slightly. This time, fixing the checkerboard colouring for the diagram of L so that the distinguished crossing is again of type II, the incidence is $\mu = -1$ (see Figure

2.5).

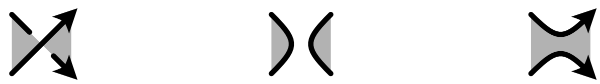


Figure 2.5 Colouring conventions for case 3: L, L_0 (the unoriented resolution) and L_1 (the oriented resolution) at the resolved negative crossing. For case 4 the white and black regions are exchanged to yield the dual colouring.

Following the conventions above, we have that

$$G = \begin{pmatrix} a & v \\ v^T & G_0 \end{pmatrix} \quad \text{and} \quad G_1 = \begin{pmatrix} a+1 & v \\ v^T & G_0 \end{pmatrix}$$

(notice that the resolutions exchange roles and have been renamed accordingly). Diagonalizing we obtain

$$\det(G) = \beta \det(G_0) \text{ and } \det(G_1) = (\beta + 1) \cdot \det(G_0)$$

so our hypothesis forces $\beta = -1$, resulting in

$$\text{signature}(G) = \text{signature}(G_0) - 1 = \text{signature}(G_1) - 1.$$

Now

$$\begin{aligned} \sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_0) - 1 - (\mu_0 + c) \\ &= \sigma_0 - c - 1 \end{aligned}$$

as in (Manolescu and Ozsváth, 2007) while

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_1) - 1 - (\mu_1 - 1) \\ &= \sigma_1.\end{aligned}$$

Finally, since

$$\widetilde{\text{Kh}}\,(L) \cong H_*\left(\widetilde{\text{Kh}}\,(L_0)\,[-\tfrac{c+1}{2}] \to \widetilde{\text{Kh}}\,(L_1)\,[\tfrac{1}{2}]\right)$$

we conclude that

$$\widetilde{\text{Kh}}\,(L)\,[-\tfrac{\sigma}{2}] \cong H_*\left(\widetilde{\text{Kh}}\,(L_0)\,[-\tfrac{\sigma_0}{2}] \to \widetilde{\text{Kh}}\,(L_1)\,[-\tfrac{\sigma_1-1}{2}]\right).$$

In terms of the σ-normalization,

$$\widetilde{\text{Kh}}_\sigma(L) = H_*\left(\widetilde{\text{Kh}}_\sigma(L_0) \to \widetilde{\text{Kh}}_\sigma(L_1)[\tfrac{1}{2}]\right)$$

as claimed.

Case 4: With distinguished negative crossing but $\det(L_0) = 0$, we use the dual colouring to that of Figure 2.5, so that the distinguished crossing is of type I with incidence $\mu = +1$, and proceed as before. In this case we have

$$G = \begin{pmatrix} a & v \\ v^T & G_1 \end{pmatrix} \quad \text{and} \quad G_0 = \begin{pmatrix} a-1 & v \\ v^T & G_1 \end{pmatrix}$$

Diagonalizing yields

$$\det(G) = \beta \det(G_1) \text{ and } \det(G_0) = (\beta - 1) \cdot \det(G_1)$$

so our hypothesis forces $\beta = +1$, resulting in

$$\text{signature}(G) = \text{signature}(G_0) + 1 = \text{signature}(G_1) + 1.$$

Therefore,

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_0) + 1 - (\mu_0 + c + 1) \\ &= \sigma_0 - c\end{aligned}$$

while

$$\begin{aligned}\sigma &= \text{signature}(G) - \mu \\ &= \text{signature}(G_1) + 1 - \mu_1 \\ &= \sigma_1 + 1\end{aligned}$$

so that

$$\widetilde{\mathrm{Kh}}\,(L) \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{c+1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[\tfrac{1}{2}]\right)$$

yields

$$\widetilde{\mathrm{Kh}}\,(L)\,[-\tfrac{\sigma}{2}] \cong H_*\left(\widetilde{\mathrm{Kh}}\,(L_0)\,[-\tfrac{\sigma_0+1}{2}] \to \widetilde{\mathrm{Kh}}\,(L_1)\,[-\tfrac{\sigma_1}{2}]\right)$$

In terms of the σ-normalization,

$$\widetilde{\mathrm{Kh}}_\sigma(L) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(L_0)[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(L_1)\right)$$

as claimed. □

CHAPTER III

HEEGAARD-FLOER HOMOLOGY

Shortly after the introduction on Khovanov homology, Ozsváth and Szabó introduced an invariant of closed, orientable 3-manifolds called Heegaard-Floer homology (Ozsváth and Szabó, 2004d; Ozsváth and Szabó, 2004c). This area has been one of intense activity, and some of the developments parallel aspects of Khovanov homology. The intention of this chapter is not to attempt a complete account of this theory, but rather a survey of those aspects that relate to this thesis' focus on Khovanov homology. Indeed, certain elements of the two theories are closely entwined, and it is on this point that we aim to elaborate.

There is a collection of notes that summarize the theory (Ozsváth and Szabó, 2006a; Ozsváth and Szabó, 2006b), as well as a survey paper (Ozsváth and Szabó, 2005a). There is also a survey by McDuff giving a slightly different perspective (McDuff, 2006), outlining in particular the role played by Lagrangian-Floer homology. Some of the most important early developments in the theory are also due (independently) to Rasmussen, and as such his work provides an excellent account. We point to (Rasmussen, 2002) and (Rasmussen, 2003), in particular.

3.1 Ozsváth and Szabó's construction

We begin by giving a brief overview of the definition of $\widehat{\mathrm{HF}}(Y)$ associated to a smooth, oriented, closed, connected 3-manifold. As with Khovanov homology, we work over

$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, and for the moment we make the further assumption that Y is a rational homology sphere.

The construction depends on a choice of *pointed* Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y, where g denotes the genus of the Heegaard surface $f^{-1}(\frac{3}{2})$ for some self indexing Morse function $f \to [0,3]$ (as in Section 1.5), and z is a point in $\Sigma_g \smallsetminus \boldsymbol{\alpha} \smallsetminus \boldsymbol{\beta}$.

The g-fold symmetric product of Σ_g is defined

$$\mathrm{Sym}^g \Sigma_g = \overbrace{\Sigma \times \cdots \times \Sigma}^{g} \Big/ S_g$$

where S_g is the symmetric group on g letters acting by permuting the coordinates. $\mathrm{Sym}^g \Sigma_g$ turns out to be a complex manifolds (see (Griffiths and Harris, 1994), for example), essentially due to the fundamental theorem of algebra.[1] For example, $\mathrm{Sym}^1 \Sigma_1 \cong S^1 \times S^1$ (obvious) and $\mathrm{Sym}^2 \Sigma_2 \cong (S^1 \times S^1 \times S^1 \times S^1)\#\overline{CP}^2$ (less obvious, see (Bertram and Thaddeus, 2001)). Symmetric products are studied extensively in (Macdonald, 1962).

Perutz demonstrates that $\mathrm{Sym}^g \Sigma_g$ is a symplectic manifold (Perutz, 2008, Section 7), and the two natural tori

$$\mathbf{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \cdots \times \alpha_g$$

and

$$\mathbf{T}_{\boldsymbol{\beta}} = \beta_1 \times \cdots \times \beta_g$$

are Lagrangian submanifolds. By isotopy of the surface Σ_g, we may assume that the intersection $\mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}$ is transverse. The key idea then, is to consider the Lagrangian-Floer homology $\mathrm{CF}(\mathbf{T}_{\boldsymbol{\alpha}}, \mathbf{T}_{\boldsymbol{\beta}})$ in this particular setting (Floer, 1988),[2] and show that it

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[1] In a local chart on Σ_g, the fundamental theorem of algebra allows us to move between the coefficients of a polynomial of degree g and its roots.

[2] While this invariant is not always well defined, the key observation here is that $\mathrm{Sym}^g \Sigma_g$ is a relatively simple symplectic manifold. In particular, $\pi_2(\mathrm{Sym}^g \Sigma_g)$ has relatively simple structure – in technical terms, $\mathrm{Sym}^g \Sigma_g$ is *monotone* – and as a result the chain complex $\mathrm{CF}(\mathbf{T}_{\boldsymbol{\alpha}}, \mathbf{T}_{\boldsymbol{\beta}})$ is well defined.

is an invariant of the underlying 3-manifold.

Let $\widehat{\mathrm{CF}}(Y)$ be the $\mathbb{F}$-vector space generated by the set of intersection points $\mathbf{x} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}$. The differential on the complex $\widehat{\mathrm{CF}}(Y)$ arises from counting holomorphic disks in $\mathrm{Sym}^g \Sigma_g$. This assumes a choice of complex structure on Σ_g, inducing an almost complex structure on $\mathrm{Sym}^g \Sigma_g$.

Let $D = \{z : |z| \leq 1\}$ be the standard unit disk in $\mathbb{C}$. For intersection points $\mathbf{x}, \mathbf{y} \in \mathbf{T}_\alpha \cap \mathbf{T}_\beta$ let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the homotopy classes of Whitney discs from $\mathbf{x}$ to $\mathbf{y}$. That is

$$\pi_2(\mathbf{x},\mathbf{y}) = \left\{ \phi : D \to \mathrm{Sym}^g \Sigma_g \,\middle|\, \begin{array}{l} \phi(-i) = \mathbf{x} \\ \phi(i) = \mathbf{y} \\ \phi(e^+) \subset \mathbf{T}_{\boldsymbol{\alpha}} \\ \phi(e^-) \subset \mathbf{T}_{\boldsymbol{\beta}} \end{array} \right\}$$

where $e^+ = \{z \in \partial D : \Re(z) > 0\}$ and $e^- = \{z \in \partial D : \Re(z) < 0\}$.

When ϕ admits a holomorphic representative, we denote the Maslov index of ϕ by $\mu(\phi)$; this quantity can be shown to be the expected dimension of the moduli space $\mathcal{M}(\phi)$ of holomorphic disks ϕ. There is a natural $\mathbb{R}$ action on D fixing $\pm i$ so that according to Gromov, $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$ is a finite number of points whenever $\mu(\phi) = 1$ (Gromov, 1985). Now the differential is defined by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \left| \widehat{\mathcal{M}}(\phi) \right|_{(\mathrm{mod}\, 2)} \mathbf{y}.$$

Here, $n_z(\phi)$ is the algebraic intersection with the complex codimension 1 submanifold

$$\{z\} \times \mathrm{Sym}^{g-1} \Sigma_g \subset \mathrm{Sym}^g \Sigma_g.$$

The definition of ∂ depends on a variety of choices which we have glossed over. In particular, a choice of complex structure on Σ_g is required, as well as a path of nearly symmetric almost complex structures on $\mathrm{Sym}^g \Sigma_g$ (see (Ozsváth and Szabó, 2004d, Section 3.1; Section 4.1)).

Theorem 3.1. *(Ozsváth and Szabó, 2004d) There exist generic choices so that* $\partial^2 = 0$*.*

Definition 3.2. *Denote by* $\widehat{\mathrm{HF}}(Y)$ *the homology of the complex* $(\widehat{\mathrm{CF}}(Y), \partial)$*.*

Theorem 3.3. *(Ozsváth and Szabó, 2004d) The homology* $\widehat{\mathrm{HF}}(Y)$ *is an invariant of the manifold* Y *specified by the pointed Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$*.*

Remark 3.4. *The proof of invariance requires an analogue of Singer's result (see Section 1.5) for pointed Heegaard diagrams, described in (Ozsváth and Szabó, 2004d, Section 7).*

There are some technical complications that arise when the restriction to rational homology spheres is removed. This is handled by considering a special subclass of *admissible* pointed Heegaard diagrams (Ozsváth and Szabó, 2004d, Section 4.2). With this done, the Heegaard-Floer homology groups are defined as above.

3.2 Variants

There is a variant $\mathrm{CF}^\infty(Y)$ that is given by a free $\mathbb{F}[U, U^{-1}]$-module generated, once again, by intersections points $\mathbf{x} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}$. In this setting,

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \left| \widehat{\mathcal{M}}(\phi) \right|_{(\mathrm{mod}\ 2)} U^{n_z(\phi)} \mathbf{y}$$

where $\deg(U) = -2$.

Using the identification $U^i \mathbf{x} = [\mathbf{x}, i]$, there is a natural subcomplex

$$\iota\colon\ \mathrm{CF}^-(Y) \hookrightarrow \mathrm{CF}^\infty(Y)$$

generated by $[\mathbf{x}, i]$ for $i \leq 0$. This is a free $\mathbb{F}[U]$-module, giving rise to a short exact sequence

$$0 \longrightarrow \mathrm{CF}^-(Y) \longrightarrow \mathrm{CF}^\infty(Y) \longrightarrow \mathrm{CF}^+(Y) \longrightarrow 0$$

The induced action $U : \mathrm{CF}^+(Y) \to \mathrm{CF}^+(Y)$ gives rise to a second short exact sequence

$$0 \longrightarrow \widehat{\mathrm{CF}}(Y) \longrightarrow \mathrm{CF}^+(Y) \longrightarrow \mathrm{CF}^+(Y) \longrightarrow 0$$

Both short exact sequences induce long exact sequences between the resulting homology groups denoted $\mathrm{HF}^\infty(Y)$, $\mathrm{HF}^-(Y)$ and $\mathrm{HF}^+(Y)$, with $\widehat{\mathrm{HF}}(Y)$ as above.

Definition 3.5. *The reduced Heegaard-Floer homology is the finitely generated* $\mathbb{F}$*-vector space given by* $\mathrm{HF}_{\mathrm{red}}(Y) = \ker(\iota_*)$*. Equivalently,* $\mathrm{HF}_{\mathrm{red}}(Y) = \ker(U^N) \subset \mathrm{HF}^-(Y)$ *for sufficiently large* N*.*

3.3 Gradings

There are two gradings on $\widehat{\mathrm{CF}}(Y)$ the first is a relative $\mathbb{Z}/2\mathbb{Z}$-grading that is switched by the differential, and the second is a splitting

$$\widehat{\mathrm{CF}}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c} \widehat{\mathrm{CF}}(Y, \mathfrak{s})$$

that descends to a splitting of $\widehat{\mathrm{HF}}(Y)$. Both gradings make use of the isomorphism

$$H_1(Y;\mathbb{Z}) \cong \frac{H_1(\Sigma_g;\mathbb{Z})}{[\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g]} \cong \frac{H_1(\mathrm{Sym}^g \Sigma_g;\mathbb{Z})}{H_1(\mathbf{T}_{\boldsymbol{\alpha}};\mathbb{Z}) \oplus H_1(\mathbf{T}_{\boldsymbol{\beta}};\mathbb{Z})} \tag{3.1}$$

The $\mathbb{Z}/2\mathbb{Z}$-grading may be seen in terms of homological data. Fixing an arbitrary orientation on both $\mathbf{T}_{\boldsymbol{\alpha}}$ and $\mathbf{T}_{\boldsymbol{\beta}}$, we can compare this to the orientation Σ_g induces on $\mathrm{Sym}^g \Sigma_g$ in the following way: set $\iota(\mathbf{x}) = \pm 1$ depending on whether or not the orientation on $T_{\mathbf{x}} \mathrm{Sym}^g \Sigma_g$ agrees with that of $T_{\mathbf{x}}\mathbf{T}_{\boldsymbol{\alpha}} \oplus T_{\mathbf{x}}\mathbf{T}_{\boldsymbol{\beta}}$. Then the algebraic intersection is given by

$$\mathbf{T}_{\boldsymbol{\alpha}} \cdot \mathbf{T}_{\boldsymbol{\beta}} = \sum_{\mathbf{x} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}} \iota(\mathbf{x}),$$

and only the overall sign on $\mathbf{T}_{\boldsymbol{\alpha}} \cdot \mathbf{T}_{\boldsymbol{\beta}}$ depends on the arbitrary choice of orientations on $\mathbf{T}_{\boldsymbol{\alpha}}$ and $\mathbf{T}_{\boldsymbol{\beta}}$.

The same intersection may be defined directly on the Heegaard surface, since an arbitrary orientation of the α_i and β_i induces an orientation on $\mathbf{T}_{\boldsymbol{\alpha}}$ and $\mathbf{T}_{\boldsymbol{\beta}}$, respectively. Now if $g_{ij} = \alpha_i \cdot \beta_j$ then $\mathbf{T}_{\boldsymbol{\alpha}} \cdot \mathbf{T}_{\boldsymbol{\beta}} = \det(g_{ij})$. Notice however that, for the natural cell decomposition obtained from the Morse function f, the matrix (g_{ij}) defines the differential $C_2(Y;\mathbb{Z}) \to C_1(Y;\mathbb{Z})$ on the cellular homology of Y. Therefore, $\mathbf{T}_{\boldsymbol{\alpha}} \cdot \mathbf{T}_{\boldsymbol{\beta}} = \pm|H_1(Y;\mathbb{Z})|$.

Now the Heegaard-Floer complex decomposes by $\widehat{\mathrm{CF}} = \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \widehat{\mathrm{CF}}_i(Y)$ where $\iota(\mathbf{x}) = (-1)^i$. Although $\iota(\mathbf{x})$ depends on the orientation imposed on $\mathbf{T}_{\boldsymbol{\alpha}}$ and $\mathbf{T}_{\boldsymbol{\beta}}$, $\iota(\mathbf{x})\iota(\mathbf{y}) = (-1)^{\mu(\phi)}$ for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. As a result, $\partial\colon \widehat{\mathrm{CF}}_i(Y) \to \widehat{\mathrm{CF}}_{i+1}$, giving rise to a $\mathbb{Z}/2\mathbb{Z}$-grading on $\widehat{\mathrm{HF}}(Y)$. Moreover, by construction

$$\chi\widehat{\mathrm{HF}}(Y) = \mathrm{rk}\,\widehat{\mathrm{CF}}_0(Y) - \mathrm{rk}\,\widehat{\mathrm{CF}}_1(Y) = \mathbf{T}_{\boldsymbol{\alpha}} \cdot \mathbf{T}_{\boldsymbol{\beta}} = \pm|H_1(Y;\mathbb{Z})|.$$

Thus, we have sketched:

Lemma 3.6. *(Ozsváth and Szabó, 2006b, Lemma 1.6)*

$$\chi\widehat{\mathrm{HF}}(Y) = \pm|H_1(Y;\mathbb{Z})|$$

Here, and throughout, we use the convention that $|H_1(Y;\mathbb{Z})| = 0$ whenever the manifold has $H_1(Y;\mathbb{Q}) \neq 0$. In general, we will fix this grading with the choice $\chi\widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|$, as in the case of $\chi\widetilde{\mathrm{Kh}}(L)$ for $L \hookrightarrow S^3$.

Remark 3.7. *This relative $\mathbb{Z}/2\mathbb{Z}$-grading admits a lift to an absolute $\mathbb{Q}$-grading (Ozsváth and Szabó, 2003a).*

There is a further refinement by decomposing according to Spin^c-structures on Y. Such a structure is a lift of the frame bundle over Y (with structural group $SO(3)$) to a principle $U(2)$-bundle over Y. Turaev gives an equivalent definition in terms of vector fields on Y (Turaev, 1997).

Definition 3.8. *Two non-vanishing vector fields v_1 and v_2 on a 3-manifold Y are called homologous if they are homotopic on the complement of a finite collection of 3-balls in Y.*

As a result, we get an equivalence relation on non-vanishing vector fields: $v_1 \sim v_2$ if and only if v_1 and v_2 are homologous. Denoting by $\mathrm{Vect}(Y)$ the space of non-vanishing vector fields on Y, Turaev shows that

$$\mathrm{Spin}^c(Y) = \mathrm{Vect}(Y)/\sim .$$

This point of view is useful in the present context, since the Morse function f gives rise to a non-vanishing vector field on Y by the following procedure. Consider the gradient vector field ∇f on Y (for some fixed Riemannian metric). By our choice of Morse function this has $2g+2$ critical points. For a given $\mathbf{x} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}$, we have a g tuple of points $(x_1, \ldots, x_g)$ in Σ_g, determining $g+1$ flow-lines $\gamma_{x_1}, \ldots, \gamma_{x_g}, \gamma_z$. Note that a neighbourhood of γ_z contains the index 0 and index 3 critical points, while the neighbourhoods of the γ_{x_i}, taken together, contain the index 1 and index 2 critical points. As a result, ∇f defines a non-vanishing vector field on Y once neighbourhoods of these flow lines are removed, and since each flow line contains exactly one critical point of each parity, $\deg \nabla f|_{\nu(\gamma_{x_1})} = \deg \nabla f|_{\nu(\gamma_z)} = 0$ and hence the vector field may be extended to give $\mathfrak{s}_z(\mathbf{x}) \in \mathrm{Spin}^c(Y)$ (see (Milnor, 1963)).

Now given a pair of points $\mathbf{x}, \mathbf{y} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}$, consider arcs $a \in \mathbf{T}_{\boldsymbol{\alpha}}$ and $b \in \mathbf{T}_{\boldsymbol{\beta}}$ beginning at $\mathbf{x}$ and ending at $\mathbf{y}$; denote by $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y;\mathbb{Z})$ the image of the class $[a-b]$ under the isomorphism (3.1). Ozsváth and Szabó show that $\pi_2(\mathbf{x}, \mathbf{y}) \neq \emptyset$ if and only if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$.[3] The splitting of $\widehat{\mathrm{HF}}(Y)$ according to $\mathrm{Spin}^c(Y)$ results then from the following:

[3]Strictly speaking, we should restrict to $g > 1$ at this point. There are technical difficulties that arise when $g = 1$; these are handled in (Ozsváth and Szabó, 2004d, Section 2.4).

Lemma 3.9. *(Ozsváth and Szabó, 2004d, Lemma 2.19)*

$$\mathfrak{s}_z(\mathbf{x}) - \mathfrak{s}_z(\mathbf{y}) = \mathrm{PD}[\epsilon(\mathbf{x},\mathbf{y})] \in H^2(Y;\mathbb{Z})$$

Remark 3.10. $\mathrm{Spin}^c(Y)$ *is an affine space for* $H^2(Y;\mathbb{Z})$*. For a fixed trivialization* $\tau\colon TY \to Y \times \mathbb{R}$ *we have*

$$\delta^\tau\colon\ \mathrm{Spin}^c(Y) \to H^2(Y;\mathbb{Z})$$

where $\delta^\tau(v) = v^*\mu$*. Here* v *is taken to be orthonormal (having fixed a Riemannian metric on* Y*) and gives a homeomorphism* $v\colon\ Y \to S^2$*, and* μ *is the generator of* $H^2(S^2;\mathbb{Z})$*. This turns out to be a bijection, and although it depends on* τ*, it can be shown that the difference* $\delta(v_1,v_2) = \delta^\tau(v_1) - \delta^\tau(v_2)$ *is independent of this choice. As a result,*

$$\delta(v,\cdot) : \mathrm{Spin}^c(Y) \to H^2(Y;\mathbb{Z})$$

gives a bijection for any $v \in \mathrm{Spin}^c(Y)$*, and writing*

$$\begin{aligned} H^2(Y;\mathbb{Z}) \times \mathrm{Spin}^c(Y) &\to \mathrm{Spin}^c(Y) \\ (a,v) &\mapsto a+v \end{aligned}$$

such that $\delta(a+v,v) = a$ *gives the affine structure. Details are spelled out in (Ozsváth and Szabó, 2004d, Section 2.6).*

Now we have that

$$\widehat{\mathrm{HF}}(Y) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} \widehat{\mathrm{HF}}(Y,\mathfrak{s}),$$

and this splitting respects the $\mathbb{Z}/2\mathbb{Z}$-grading. As a result, Lemma 3.6 may be refined:

Lemma 3.11. *(Ozsváth and Szabó, 2004c, Proposition 5.1)*

$$\chi\widehat{\mathrm{HF}}(Y,\mathfrak{s}) = \begin{cases} \pm 1 & \text{if } H_1(Y;\mathbb{Q}) = 0 \\ 0 & \text{otherwise} \end{cases}$$

3.4 The surgery exact sequence

There is a long exact sequence in Heegaard-Floer homology that results from considering surgery on a knot, or more generally fillings of a knot manifold. Given a manifold M with torus boundary, together with a pair of slopes α and β forming a basis for surgery (that is , $\alpha \cdot \beta = +1$), then the triple of manifolds $(M(\alpha), M(\beta), M(\alpha+\beta))$ form a *triad* of 3-manifolds; the triple $(\alpha, \beta, \alpha+\beta)$ is a triad of slopes.

The key property enjoyed by a triad is as follows. Note that, given a choice of orientation on the rational longitude λ_M, if there exists choices[4] for which $\alpha \cdot \lambda_M = +1$ and $\beta \cdot \lambda_M = +1$ we have

$$\begin{aligned} |H_1(M(\alpha+\beta);\mathbb{Z})| &= c_M \Delta(\alpha+\beta, \lambda_M) \\ &= c_M |\alpha \cdot \lambda_M + \beta \cdot \lambda_M| \\ &= c_M |\alpha \cdot \lambda_M| + c_M |\beta \cdot \lambda_M| \\ &= |H_1(M(\alpha);\mathbb{Z})| + |H_1(M(\beta);\mathbb{Z})| \end{aligned}$$

Now there is a long exact sequence relating any such triad

$$\longrightarrow \widehat{\mathrm{HF}}(M(\alpha)) \longrightarrow \widehat{\mathrm{HF}}(M(\beta)) \longrightarrow \widehat{\mathrm{HF}}(M(\alpha+\beta)) \longrightarrow$$

Notice that this relates +1-surgery on a β-framed knot in $M(\alpha)$, to the manifolds $M(\alpha)$ and $M(\beta)$, hence the terminology *surgery* exact sequence. In particular, for a knot $K \hookrightarrow S^3$ we have that

$$\longrightarrow \widehat{\mathrm{HF}}(S^3) \longrightarrow \widehat{\mathrm{HF}}(S^3_n(K)) \longrightarrow \widehat{\mathrm{HF}}(S^3_{n+1}(K)) \longrightarrow$$

for any $n \geq 0$.

With this exact sequence as a point of departure, Ozsváth and Szabó demonstrate an

[4] Such choices always exist, though this generally comes at the expense $\alpha \cdot \beta = \pm 1$.

incredible relationship between Khovanov homology and Heegaard-Floer homology.

Theorem 3.12. *(Ozsváth and Szabó, 2005c, Theorem 1.1) There is a spectral sequence with* $E_2 \cong \widetilde{\mathrm{Kh}}(L^\star)$*, converging to* $E_\infty \cong \widehat{\mathrm{HF}}(\mathbf{\Sigma}(S^3, L))$*, where* $L^\star$ *denotes the mirror image of* L*.*

Recall that $\widetilde{\mathrm{Kh}}(L^\star)$ amounts to considering the dual of $\widetilde{\mathrm{Kh}}(L)$. We will elaborate on aspects of this result in the next chapter, recording for the moment the following:

Corollary 3.13. *(Ozsváth and Szabó, 2005c, Corollary 1.2)*

$$\det(L) \leq \operatorname{rk} \widehat{\mathrm{HF}}(\mathbf{\Sigma}(S^3, L)) \leq \operatorname{rk} \widetilde{\mathrm{Kh}}(L)$$

Proof. The first inequality follows from $\det(L) = |H_1(\mathbf{\Sigma}(S^3, L); \mathbb{Z})|$, together with Lemma 3.6. The second inequality results from $\operatorname{rk} \widetilde{\mathrm{Kh}}(L) = \operatorname{rk} \widetilde{\mathrm{Kh}}(L^\star)$, together with Theorem 3.12. □

Of course, we have observed previously that $\det(L) = \chi \widetilde{\mathrm{Kh}}(L)$, which yields the inequality $\det(L) \leq \operatorname{rk} \widetilde{\mathrm{Kh}}(L)$ (see Section 2.4).

3.5 L-spaces

An L-space is a rational homology sphere with Heegaard-Floer homology that has smallest possible rank. The prototypical examples are lens spaces,[5] and in particular S^3 is an L-space.

Definition 3.14. *A closed, connected, orientable 3-manifold is an L-space if it is a rational homology sphere with the property that*

$$\operatorname{rk} \widehat{\mathrm{HF}}(Y) = |H_1(Y; \mathbb{Z})| \,.$$

[5]Hence, L-space abbreviates the somewhat longer moniker *Heegaard-Floer homology lens space.*

Equivalently, these manifolds are characterized by having $\mathrm{HF}_{\mathrm{red}}(Y) = 0$. While L-spaces are certainly of interest in the context of Heegaard-Floer homology, they seem to be an important class of manifolds more generally.

Theorem 3.15. *(Ozsváth and Szabó, 2004a, Theorem 1.4) L-spaces do not admit taut foliations.*[6]

We devote this section to some interesting examples of L-spaces (see (Ozsváth and Szabó, 2005b; Ozsváth and Szabó, 2005c)).

Proposition 3.16. $\mathbf{\Sigma}(S^3, L)$ *is an L-space whenever* L *is a homologically thin link.*

Proof. This is immediate from Corollary 3.13, combined with the fact that $det(L) = \operatorname{rk} \widetilde{\mathrm{Kh}}(L)$ for thin links (see Section 2.4). $\square$

Since non-split, alternating links[7] are thin (Lee, 2005), it follows that the two-fold branched cover of a non-split, alternating link is an L-space. These links are a subset of a much larger class with the same property.

Definition 3.17. *The set of quasi-alternating links* $\mathcal{Q}$ *is the smallest set of links containing the trivial knot, and closed under the following relation: if* L *admits a projection with distinguished crossing* $L(\times)$ *so that*

$$\det(L(\times)) = \det(L(\asymp)) + \det(L()\,())$$

for which $L(\asymp), L()\,() \in \mathcal{Q}$*, then* $L = L(\times) \in \mathcal{Q}$ *as well.*

Ozsváth and Szabó show that non-split, alternating links are quasi-alternating, and that $\mathbf{\Sigma}(S^3, L)$ is an L-space whenever L is quasi-alternating (Ozsváth and Szabó, 2005c).

[6] In this context, a foliation $\mathcal{F}$ of Y is called taut whenever it is co-orientable, and there exists a closed curve in Y that meets every leaf of $\mathcal{F}$ transversally (Eliashberg and Thurston, 1998).

[7] Recall that, by definition, an alternating link admits an alternating link diagram.

Indeed, this may be seen as a generalization of Lee's result, as Manolescu and Ozsváth have shown that quasi-alternating links are homologically thin (Manolescu and Ozsváth, 2007). We remark however that $w(L) = 1$ (see Definition 2.10) is not equivalent to $L \in \mathcal{Q}$: examples of homologically thin knots that are not quasi-alternating have been given by A. Shumakovitch[8] and J. Greene[9].

Proposition 3.18. *(Ozsváth and Szabó, 2005b, Proposition 2.3) A manifold with elliptic geometry (equivalently, finite fundamental group, see Remark 1.17) is an L-space.*

In fact, there is a complete characterization of Seifert fibered L-spaces (in terms of Seifert invariants) whenever the base orbifold is S^2 (Ozsváth and Szabó, 2003c).

As a particular example, the Poincaré homology sphere is an L-space, although this manifold (and its mirror image) is the only known prime, integer homology three-sphere with this property.

Question 3.19. *Are the Poincaré homology sphere, its mirror image, and S^3 the only prime manifolds for which the Heegaard-Floer homology is rank one?* [10]

Remark 3.20. *This example demonstrates, however, that $w(L) = 1$ is not necessary to obtain an L-space $\mathbf{\Sigma}(S^3, L)$: the Poincaré homology sphere arises as $\mathbf{\Sigma}(S^3, 10_{124})$ (see Chapter 1) where $w(10_{124}) = 2$ (see Chapter 2).*

From the surgery point of view, L-spaces are somewhat rare. For example:

Theorem 3.21. *(Ozsváth and Szabó, 2005b, Theorem 1.2) If $K \hookrightarrow S^3$ yields an L-space*

[8] In a remark during a lecture by C. Manolescu at the conference Knots in Washington XXVI: the knot 9_{46} has thin Khovanov homology but an off-diagonal $\mathbb{Z}/3\mathbb{Z}$ in odd-Khovanov homology (Ozsváth et al., 2007).

[9] Private communication: the knot 11^n_{50} has Khovanov, odd-Khovanov, and knot Floer homologies all supported in a single diagonal but it is not quasi-alternating.

[10] A conjecture has not been made, in print, in either direction. However, during his lectures at PCMI in 2006, Z. Szabó conjectured that the answer is "yes".

via Dehn surgery, then

$$\Delta_K(t) = (-1)^k + \sum_{i=1}^{k} (-1)^{k-i}(t^{n_i} + t^{-n_i})$$

where $\Delta_K(t)$ *is the Alexander polynomial of* K*, for some sequence of integers* $0 < n_1 < \cdots < n_k$.

This is quite restrictive, as demonstrated for example by (the proof of) the following fact:

Theorem 3.22. *(Ozsváth and Szabó, 2005b, Theorem 1.5) Surgery on a hyperbolic, alternating knot in* S^3 *never yields an L-space.*

Further restrictions are given by the topology of the knot complement.

Theorem 3.23. *(Ghiggini, 2008; Ni, 2007) If* $K \hookrightarrow S^3$ *admits an L-space via Dehn surgery, then* K *must be fibered.*

Theorem 3.24. *(Kronheimer et al., 2007; Hedden, 2007; Rasmussen, 2007) If* $S^3_n(K)$ *is an L-space, then* $g \leq n$ *where* g *is the Seifert genus of* K.

On the other hand, a given L-space surgery on S^3 yields an infinite family of L-spaces.

Proposition 3.25. *(Ozsváth and Szabó) For any triad of 3-manifolds* $(M(\alpha), M(\beta), M(\alpha+\beta))$*, if* $M(\alpha)$ *and* $M(\beta)$ *are L-spaces, then* $M(\alpha+\beta)$ *is an L-space as well.*

Proof. Combining the surgery exact sequence with the homological properties of the triad we obtain

$$\mathrm{rk}\,\widehat{\mathrm{HF}}(M(\alpha+\beta)) \leq \mathrm{rk}\,\widehat{\mathrm{HF}}(M(\alpha)) + \mathrm{rk}\,\widehat{\mathrm{HF}}(M(\beta)) = |H_1(M(\alpha+\beta);\mathbb{Z})|$$

□

It follows that if $S^3_n(K)$ is an L-space, then so is $S^3_{n+1}(K)$. More generally we have:

Proposition 3.26. *(Ozsváth and Szabó) Given a knot* $K \hookrightarrow S^3$ *for which* $S^3_{p/q}(K)$ *is an L-space,* $S^3_{r/s}(K)$ *is also an L-space for all* $\frac{r}{s} \geq \frac{p}{q}$*.*

Proof. We include a very quick – though machinery heavy – proof of this fact. Calculating the rank of the Heegaard-Floer homology for surgery on a knot $K \hookrightarrow S^3$ is accomplished by the formula

$$\operatorname{rk} \widehat{\mathrm{HF}}(S^3_{p/q}(K)) = |p| + 2\max\{0, (2B_K - 1)|q| - |p|\} + |q|C_K$$

from (Ozsváth and Szabó, 2005d, Proposition 9.5), for non-negative constants B_K, C_K depending on K. In our setting, we may assume that $p, q > 0$, and since $S^3_{p/q}(K)$ is an L-space, $\operatorname{rk} \widehat{\mathrm{HF}}(S^3_{p/q}(K)) = p$. This forces $(2B_K - 1)q - p \leq 0$ and $C_K = 0$.

Now suppose $\frac{r}{s} \geq \frac{p}{q}$. Then

$$\operatorname{rk} \widehat{\mathrm{HF}}(S^3_{r/s}(K)) = r + 2\max\{0, (2B_K - 1)s - r\}$$

but

$$B_K \leq \tfrac{1}{2}(\tfrac{p}{q} + 1) \leq \tfrac{1}{2}(\tfrac{r}{s} + 1)$$

forces $(2B_K - 1)s - r \leq 0$ so that $\operatorname{rk} \widehat{\mathrm{HF}}(S^3_{r/s}(K)) = r$ as claimed. □

Thus, despite the fact that L-spaces seem to be rare in certain respects, it is easy to construct large families of L-spaces:

Corollary 3.27. *Up to taking mirrors, all sufficiently large surgeries on a torus knot (or more generally, Berge knot) yield L-spaces.*

Another interesting family of examples results from considering certain pretzel knots.

Theorem 3.28. *(Goda et al., 2005; Ozsváth and Szabó, 2005b) The* $(-2, 3, q)$*-pretzel knots admit L-space surgeries for all* $q \geq 3$*.*

This results from the calculation of the knot Floer homology (see Section 3.7) for this family of knots (Goda et al., 2005, Theorem 5.1), together with (Ozsváth and Szabó, 2005b, Theorem 1.3). See also (Ozsváth and Szabó, 2005b, Page 12). Of course, for $q = 3, 5$ these are torus knots, when $q = 7$ this pretzel is a Berge knot (in fact, an example of a hyperbolic knot admitting a lens space surgery of (Fintushel and Stern, 1980)), and when $q = 9$ we obtain an example of Bleiler and Hodgson of a hyperbolic knot admitting finite fillings (Bleiler and Hodgson, 1996).

The large-surgery property (Proposition 3.26) for L-spaces gives rise to another interesting class:

Proposition 3.29. *(Boyer and Watson, 2009) Suppose Y is a Seifert fibered space with base orbifold $\mathcal{B} = RP^2(a_1, \ldots, a_n)$. Then Y is an L-space.*

Proof. First recall that if $\mathcal{B} = RP^2(a_1)$ then the Seifert structure is not unique. Such a Y is either $RP^3\#RP^3$ or admits a Seifert fibre structure with base orbifold $S^2(2, 2, n)$ for some $n > 0$. Note however that Y has finite fundamental group in this case (see Proposition 1.16), and is therefore an L-space according to Proposition 3.18.

We take this as a base case for induction on the number of singular fibres. Suppose that any Y with base orbifold $RP^2(a_1, \ldots, a_n)$ is an L-space. Choose a regular fibre φ in Y and let $M = Y \smallsetminus \nu(\varphi)$. This is a manifold with torus boundary for which $H_1(M; \mathbb{Q}) = \mathbb{Q}$ (see Section 1.7). The rational longitude λ_M coincides (as a slope in ∂M) with a regular fibre in ∂M according to Proposition 1.22.

Choosing a meridian μ for the fibre φ with the property that $\mu \cdot \lambda_M = 1$ we have a basis for Dehn surgery. That is

$$Y_{p/q}(\varphi) = M(\alpha)$$

where $\alpha = p\mu + q\lambda_M$. Note that this new Seifert fibered space has base orbifold

$$RP^2(a_1, \ldots, a_n, a_{n+1})$$

where $a_{n+1} = \Delta(\alpha, \lambda_M)$ by Theorem 1.21. In particular, by our induction hypothesis $M(\alpha)$ is an L-space whenever $\Delta(\alpha, \lambda_M) = 1$. This occurs whenever $\alpha = \mu + q\lambda_M$ for any $q \in \mathbb{Z}$.

Now $(\mu, \mu + \lambda_M, 2\mu + \lambda_M)$ form a triad of slopes in ∂M. Since $\mu \cdot \lambda_M = 1$, this follows readily from the fact that

$$\begin{aligned} |H_1(M(2\mu + \lambda_M); \mathbb{Z})| &= c_M \Delta(2\mu + \lambda_M, \lambda_M) \\ &= c_M \Delta(\mu, \lambda_M) + c_M \Delta(\mu + \lambda_M, \lambda_M) \\ &= |H_1(M(\mu); \mathbb{Z})| + |H_1(M(\mu + \lambda_M); \mathbb{Z})| \end{aligned}$$

where $c_M > 0$ is a fixed constant depending only on M as in Lemma 1.5. As a result $Y_2(\varphi) = M(2\mu + \lambda_M)$ is an L-space, and moreover $Y_n(\varphi) = M(n\mu + \lambda_M)$ is an L-space for all $n > 0$, since both $M(\mu)$ and $M(\mu + \lambda_M)$ are L-spaces.

This observation does not depend on our choice of μ, and more generally, given $\alpha = \mu + q\lambda_M$ for any integer q, the triple $(\alpha, n\alpha + \lambda_M, (n+1)\alpha + \lambda_M)$ form a triad for any $n > 0$. This completes the induction, as we have that $Y_{p/q}(\varphi)$ is an L-space for every p, q with $p, q \in \mathbb{Z}$ for which $(p, q) = 1$ and $p > 0$.[11] In other words, $M(\alpha)$ is an L-space for any slope $\alpha \neq \lambda_M$ (that is, any slope other than the fibre slope). Of course, $H_1(M(\lambda_M); \mathbb{Q}) = \mathbb{Q}$, so this manifold cannot be an L-space. $\square$

With these properties and examples in hand, consider the following open problem:

Question 3.30. *(Ozsváth and Szabó, 2005a, Question 11) Is there a topological classification on L-spaces (that is, one that does not reference Heegaard-Floer homology)?*

[11] Indeed, for any slope $p\mu + q\lambda_M$, writing $\mu = \alpha - q'\lambda_M$ for some q' we have that $p\mu + q\lambda_M = p\alpha + (q - pq')\lambda_M$ in terms of the basis (α, λ_M).

3.6 A characterization of Seifert fibered L-spaces

Definition 3.31. *A group G is called left-orderable if there exists a strict total ordering $<$ on its elements such that $g < h$ implies $fg < fh$ for all elements $f, g, h \in G$.*

While the trivial group obviously satisfies such a criteria, for the present purposes we will fix the convention that the trivial group is *not* left-orderable. By a result of Howie and Short, any manifold M with torus boundary satisfying $H_1(M;\mathbb{Q}) = \mathbb{Q}$ gives an example of a fundamental group that is left-orderable (Howie and Short, 1985). However, it is certainly possible that Dehn filling of such a manifold yields a manifold with fundamental group that is not left-orderable, and this phenomenon has been studied extensively in work of Boyer, Rolfsen and Wiest (Boyer et al., 2005).

The aim of this section is to establish a connection between L-spaces and orderablity of fundamental groups.

Theorem 3.32. *(Boyer and Watson, 2009) Suppose Y is a closed, connected, orientable, Seifert fibered 3-manifold. Then Y is an L-space if and only if $\pi_1(Y)$ is not left-orderable.*

Proof. If Y is a rational homology sphere then the base orbifold has underlying surface either S^2 or RP^2 (see Section 1.7).

By a result of Lisca and Stipsicz (Lisca and Stipsicz, 2007, Theorem 1.1), in the case where the base orbifold is S^2, Y is an L-space if and only if Y does not admit a horizontal foliation. By a result of Boyer, Rolfsen and Wiest (Boyer et al., 2005, Theorem 1.3(b)), these Y admit a horizontal foliation if and only if $\pi_1(Y)$ is left-orderable.

The result of (Boyer et al., 2005, Theorem 1.3(b)) does not restrict to the case $B = S^2$, and indeed if $B = RP^2$ then $\pi_1(Y)$ is never left-orderable (unless, of course, $H_1(Y;\mathbb{Q}) \neq 0$). Thus, to conclude the proof we appeal to Proposition 3.29. □

Remark 3.33. *As noted previously, Ozsváth and Szabó give a characterization of Seifert* *fibered L-spaces (in terms of Seifert invariants) whenever the base orbifold is S^2 (Ozsváth and Szabó, 2003c). This, in turn, is exploited in (Lisca and Stipsicz, 2007), and leads to the above result when the base orbifold is orientable when combined with (Boyer et al., 2005).*

3.7 The knot filtration

There is a refinement of Heegaard-Floer homology to an invariant for knots in S^3 (more generally, to rationally null homologous knots in an arbitrary 3-manifold). This arises from the fact that the knot induces a filtration on the Heegaard-Floer homology of the underlying 3-manifold; this filtration controls the Heegaard-Floer homology of manifolds obtained by surgery on the knot, a fact discovered independently in (Ozsváth and Szabó, 2004b) and (Rasmussen, 2003). This is a powerful tool, and is the source of results such as Theorem 3.21, as well as machinery such as that used in the proof of Proposition 3.26. Indeed, the knot filtration gives rise to a mapping cone formula for computing the Heegaard-Floer homology groups resulting from surgery (Ozsváth and Szabó, 2008; Ozsváth and Szabó, 2005d).

A knot in S^3 may be described by specifying a *doubly* pointed Heegaard diagram for S^3, $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$. This means that S^3 decomposes along Σ_g according to some Morse function $f : S^3 \to [0,3]$, and the union of the gradient flow lines specified by z and w form a knot K (passing through the index 0 and index 3 critical points of f).

Now $\widehat{\mathrm{CF}}(S^3)$ may be described using the pointed Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, and the knot K specified by introducing the second point w induces a filtration on the complex

$$\mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{y}) = n_w(\phi) - n_z(\phi)$$

for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. Notice that if $\mathbf{y}$ appears in $\partial\mathbf{x}$ then $\mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{y}) \geq 0$ since $n_z(\phi) = 0$, defining a subcomplex

$$\mathcal{F}(K, i) \subset \widehat{\mathrm{CF}}(S^3).$$

Considering the induced homomorphism $H_*(\mathcal{F}(K,i)) \to \widehat{\mathrm{HF}}(S^3) \cong \mathbb{F}$ gives rise to a knot invariant $\tau(K)$, defined as the smallest integer i for which this morphism is non-trivial. In general, $|\tau(K)|$ gives a lower bound on the Seifert genus, but whenever K admits lens space surgery we have that $|\tau(K)| = g(K)$ (Ozsváth and Szabó, 2003b; Ozsváth and Szabó, 2005b).

The homology of the associated graded quotient complex defines the knot Floer homology

$$\widehat{\mathrm{HFK}}(S^3, K, i) \cong H_* \left(\mathcal{F}(K,i)/\mathcal{F}(K,i-1) \right).$$

There is some information lost in passing to the homology of the associated graded quotient complex, but this still yields a powerful invariant. It may be computed by defining $\widehat{\mathrm{CFK}}(S^3, K)$ the $\mathbb{F}$-vector space generated by $\mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}$ as usual, but imposing the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbf{T}_{\boldsymbol{\alpha}} \cap \mathbf{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1 \\ n_z(\phi)=0, n_w(\phi)=0}} \left| \widehat{\mathcal{M}}(\phi) \right|_{(\mathrm{mod}\ 2)} \mathbf{y}.$$

Writing $\widehat{\mathrm{HFK}}(S^3, K, i) = \widehat{\mathrm{HFK}}(K, i)$ we have that this theory categorifies the Alexander polynomial in the sense that

$$\Delta_K(t) = \sum (-1)^u t^i \operatorname{rk} \widehat{\mathrm{HFK}}_u(K, i).$$

While the similarity here to Khovanov homology is striking, it is particularly intriguing given that the constructions of each of these invariants is extremely different.

3.8 Characterizations of the trivial knot

As an invariant of knots in S^3, knot Floer homology has the following notable property.

Theorem 3.34. *(Ozsváth and Szabó, 2004a, Theorem 1.2) Let g be the Seifert genus of a knot $K \hookrightarrow S^3$. Then $\widehat{\mathrm{HFK}}(K, g) \neq 0$, and in particular this invariant detects the trivial knot.*

There are some weaker incarnations of this fact that will be useful, characterizing the trivial knot in terms of surgery.

Proposition 3.35. *If $S^3_n(K)$ is an L-space for every $n \neq 0$ then K is the trivial knot.*

Proof. If $S^3_n(K)$ is an L-space, for all $n \neq 0$, then in particular $S^3_{+1}(K)$ is an L-space. Then by Theorem 3.24, $g \leq n$, and if K is non-trivial we may assume that $g = 1$. Hence by Theorem 3.23, K is a genus 1 fibered knot and must be the trefoil. In fact, $S^3_{+1}(K)$ must be the Poincaré sphere, and K is the right-hand trefoil.

Now consider $S^3_{-1}(K)$: this manifold must also be an L-space by our hypothesis. However, it is well known that -1-surgery on the right-hand trefoil yields the same manifold as the $+1$-surgery on the figure eight knot (see, for example, (Rolfsen, 1976, Chapter 9)). But this contradicts Theorem 3.22 as this knot is alternating but not torus, hence does not admit L-space surgeries.[12] □

It is interesting to note that Proposition 3.35 is true only when restricting to knots in S^3: Proposition 3.29 shows that any regular fibre in a Seifert fibration over RP^2 is a knot with this same property.

Proposition 3.36. *If $S^3_N(K)$ is an L-space, for all N large enough in absolute value, then K is the trivial knot.*

Proof. Since $S^3_N(K)$ is an L-space for $N >> 0$ we have that $g(K) = \tau(K)$ by (Ozsváth and Szabó, 2005b, Proposition 3.3). On the other hand, $S^3_{-N}(K) \cong -S^3_N(K^\star)$ is an L-space as well, so that $g(K^\star) = \tau(K^\star)$. However, it is a standard property of τ that $\tau(K^\star) = -\tau(K)$ (Ozsváth and Szabó, 2003b, Lemma 3.3). Therefore, since $g(K) = g(K^\star)$ we have shown that $\tau(K) = g(K) = -\tau(K)$ hence $g(K) = 0$ and K must be the trivial knot. □

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[12] Equivalently, it may be seen by direct computation via the mapping cone formula for integer surgeries (Ozsváth and Szabó, 2008) that -1-surgery on the right-hand trefoil is not an L-space. Note also that the calculation of $\widehat{\mathrm{HF}}(S^3_n(K))$, when K is the trefoil, was originally given in (Ozsváth and Szabó, 2004c). In brief, Proposition 3.35 is certainly “known to the experts”.

We remark that, from the argument above, it is enough to have the existence of some N_0 for which both $S^3_{+N_0}(K)$ and $S^3_{-N_0}(K)$ are L-spaces to ensure that K is the trivial knot.

Knots in S^3 are also well understood in the context of Question 3.19 (see, for example, (Hedden and Watson, 2008)):

Proposition 3.37. *If $S^3_{1/q}(K)$ is an L-space for some non-trivial knot K, then $q = 1$ (respectively -1) and K is the right-hand (respectively left-hand) trefoil. In particular, the Poincaré homology sphere (and its mirror image) are the only non-trivial L-space integer homology spheres that arise via surgery on a knot in S^3.*

Proof. By passing to the mirror image of K if necessary, we may assume without loss of generality that $q > 0$.

Since $S^3_{1/q}(K)$ is an L-space, Proposition 3.26 ensures that $S^3_{+1}(K)$ is an L-space as well. In this case, Theorem 3.24 forces $g \leq 1$, and since K is non-trivial by hypothesis we have that $g = 1$ (and the knot Floer homology of K must be that of the trefoil by (Ozsváth and Szabó, 2005b)). Now Theorem 3.23 implies that K is fibered. Thus, as a fibered, genus one knot admitting an L-space surgery, K can only be the trefoil. □

CHAPTER IV

INVOLUTIONS AND TANGLES

We turn now to tangles, one of our primary objects of study. These arise naturally as the component pieces of knots and links (the approach taken by Conway in his enumeration of knots (Conway, 1970)), however we will be more interested in tangles as the branch sets for certain manifolds with torus boundary (Lickorish takes this is the point of view (Lickorish, 1981)).

It is difficult to give accurate historical references for much of this material, as many of the results seem firmly entrenched in folklore. The decomposition of knots into tangles, and in particular the relationship between *rational* tangles and continued fractions, however, is generally attributed to Conway (Conway, 1970).[1] For more on this approach, new proofs and further references see (Goldman and Kauffman, 1997; Kauffman and Lambropoulou, 2004).

The study of tangles from the point of view of two-fold branched covers seems to have been popularized by Montesinos (Montesinos, 1975). The approach taken in this work is heavily influenced by Montesinos' unpublished notes (Montesinos, 1976), as well as the work of Lickorish in the study of prime knots (Lickorish, 1981). We also point to (Bleiler, 1985; Montesinos and Whitten, 1986) bearing particular relation to this work, though these references are certainly not exhaustive.

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[1] The tangles that we will consider are 2-tangles, sometimes called Conway tangles.

Finally, much of the material that will be needed can be found in Rolfsen's classic text (Rolfsen, 1976), which has become the standard reference.

4.1 Tangles

A tangle is a pair $T = (B^3, \tau)$ where B^3 is a 3-ball and $\tau \hookrightarrow B^3$ is a pair of properly embedded arcs meeting the boundary transversally in 4 distinct points, together with a finite collection (possibly empty) of closed components. That is,

$$\tau : I \sqcup I \sqcup \underbrace{S^1 \sqcup \cdots \sqcup S^1}_{k \geq 0} \hookrightarrow B^3.$$

Equivalence of tangles is through homeomorphism of the pair (B^3, τ) that need not fix the boundary in general (though $\partial\tau$ is always 4 points). This is the point of view taken in Lickorish, for example (Lickorish, 1981).

Tangles arise naturally as component pieces of knots. Given a knot $K \hookrightarrow S^3$ and an embedding $S^2 \hookrightarrow S^3$ such that S^2 intersects K transversely in 4 points, the resulting decomposition of S^3 into 3-balls restricts to a decomposition of K into tangles, denoted $K = T_0 \cup T_1$. Since $\mathbf{\Sigma}(S^2, \{4 \text{ points}\}) = \mathbf{\Sigma}(\partial B^3, \partial\tau)$ is a torus, the key observation is that such a decomposition of K lifts to a decomposition of the two-fold branched cover $\mathbf{\Sigma}(S^3, K)$ along a torus.

A tangle is called rational whenever it is homeomorphic to the tangle $(B^3, \asymp)$. These are the simplest tangles, but they play an important role.

Definition 4.1. *A knot K has tangle unknotting number one if there is a decomposition $K = T_0 \cup T_1$ with the property that $T_0 \cup T_2$ is the trivial knot, where T_1 and T_2 are rational tangles.*

Notice that this generalizes the common notion of unknotting number one: such a knot contains the specific rational tangle $(B^3, \times)$, and becomes trivially knotted when this tangle is replaced with the tangle $(B^3, \times)$. Of course $(B^3, \times) \cong (B^3, \times)$ as tangles (in the present setting), though this is rarely the case when such a tangle is included in a knot since this effectively fixes a choice of framing (see below).

4.2 An action of the 3-strand braid group

As with knot diagrams, we will generally confuse a tangle $T = (B^3, \tau)$ and a diagram in the plane representing it. However, as we are considering tangles up to homeomorphism that need not fix the boundary, there are *many* diagrams (in the sense of Conway) representing a given tangle (in the sense of Lickorish).

To this end, we introduce a particular action of the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \big| \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

on the space of tangles, $\mathcal{T}$. Braids in this setting are depicted horizontally, read from left to right, with standard generators

$\sigma_1 =$  $\sigma_2 =$ 

For a given braid $\beta \in B_3$ the action

$$\begin{aligned} \mathcal{T} \times B_3 &\to \mathcal{T} \\ (T, \beta) &\mapsto T^\beta \end{aligned}$$

is defined by taking T^β as the tangle depicted in Figure 4.1.

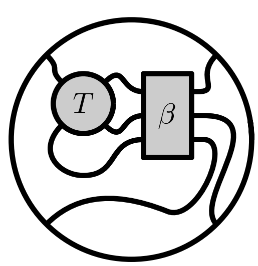


Figure 4.1 The tangle T^β.

It is straightforward to verify that this is a well defined action on tangles (see (Watson, 2006), for example). Notice that this specifies a homeomorphism of the given tangle, and as such this action is trivial when considering tangles up to homeomorphism (though the choice of diagram for a fixed tangle may be altered dramatically). However, this action of B_3 turns out to be useful when viewed as a change of framing.

4.3 Strong inversions and two-fold branched covers

A knot $K \hookrightarrow S^3$ is called strongly invertible whenever there is an involution taking the knot to itself, and fixing exactly 2 points on the knot. Since there is a unique orientation preserving involution with non-empty fixed point set on S^3 up to isotopy (Waldhausen, 1969),[2] an equivalent definition is that such knots may be put into general position with respect to the fixed point set of this involution, as follows (c.f. Definition 1.11):

Definition 4.2. *Given a knot $K \hookrightarrow S^3$, let f be the restriction of the standard involution on S^3 to the complement $M = S^3 \smallsetminus \nu(K)$. A knot $K \hookrightarrow S^3$ is strongly invertible if f is an involution on M for which* $\mathrm{Fix}(f)$ *intersects the boundary ∂M transversally in exactly 4 points.*

Notice that in this setting $\mathrm{Fix}(f)$ is always a pair of arcs embedded in M.

A natural first example is given by the trivial knot. This is a strongly invertible knot by virtue of the fact that the solid torus is a two-fold branched cover of a solid ball, branched over a pair of unknotted arcs. These arcs are obtained by intersecting a tubular neighbourhood of the trivial knot (a solid torus) with the fixed point set of the standard involution on S^3. In fact, we have the following equivalent definition of a rational tangle (see (Lickorish, 1981)).

Definition 4.3. *A rational tangle has two-fold branched cover that is a solid torus.*

The collection of rational tangles arise by considering T^β for all $\beta \in B_3$ where $T =$

[2]Thus, we take as *standard* involution on $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ the rotation fixing the z-axis.

$(B^3, \asymp)$. As a result, each of these choices of representative for T are in bijection with the possible Seifert fibrations of the solid torus (as in Section 1.7) in the cover (see (Montesinos, 1976)).

More generally, we have the following:

Proposition 4.4. *When $K \hookrightarrow S^3$ is strongly invertible, the quotient of M by the action of f is a 3-ball.*

Proof. Extending f to S^3, across the surgery torus of the trivial surgery, gives the standard involution on S^3 by definition of strong invertibility. The quotient of this involution is S^3, decomposed along a sphere obtained by the quotient of the torus ∂M. Since S^3 decomposes into a pair of 3-balls for any smooth embedding $S^2 \hookrightarrow S^3$, M/f must therefore be homeomorphic to B^3. $\square$

As a result, for any strongly invertible knot $K \hookrightarrow S^3$, the complement $M = S^3 \smallsetminus \nu(K)$ is a two-fold branched cover of a tangle $T = (B^3, \tau)$, where τ is given by the image of $\text{Fix}(f)$ in the quotient. Thus, while M/f is a relatively simple manifold, as an orbifold it may be quite complicated.

A second example is provided by the trefoil. This is a strongly invertible knot, as illustrated in Figure 4.2. To construct the tangle that arises as the quotient, a fundamental domain for the involution is needed. Then the tangle may be obtained by isotopy, as shown. With a little more care, it is possible to keep track of the image of the canonical longitude in the quotient (see (Bleiler, 1985), for example). The resulting tangle diagrams illustrated are homeomorphic, giving two different representatives for the quotient tangle. Indeed, by (Schreier, 1924) or (Montesinos, 1976), this tangle is unique up to homeomorphism.

Remark 4.5. *There is another way to see that the tangle given in Figure 4.2 is accurate, from the point of view of (Montesinos, 1976). Notice that this tangle is a sum of two rational tangles: this reflects the Seifert fibre structure in the cover, recalling that the*

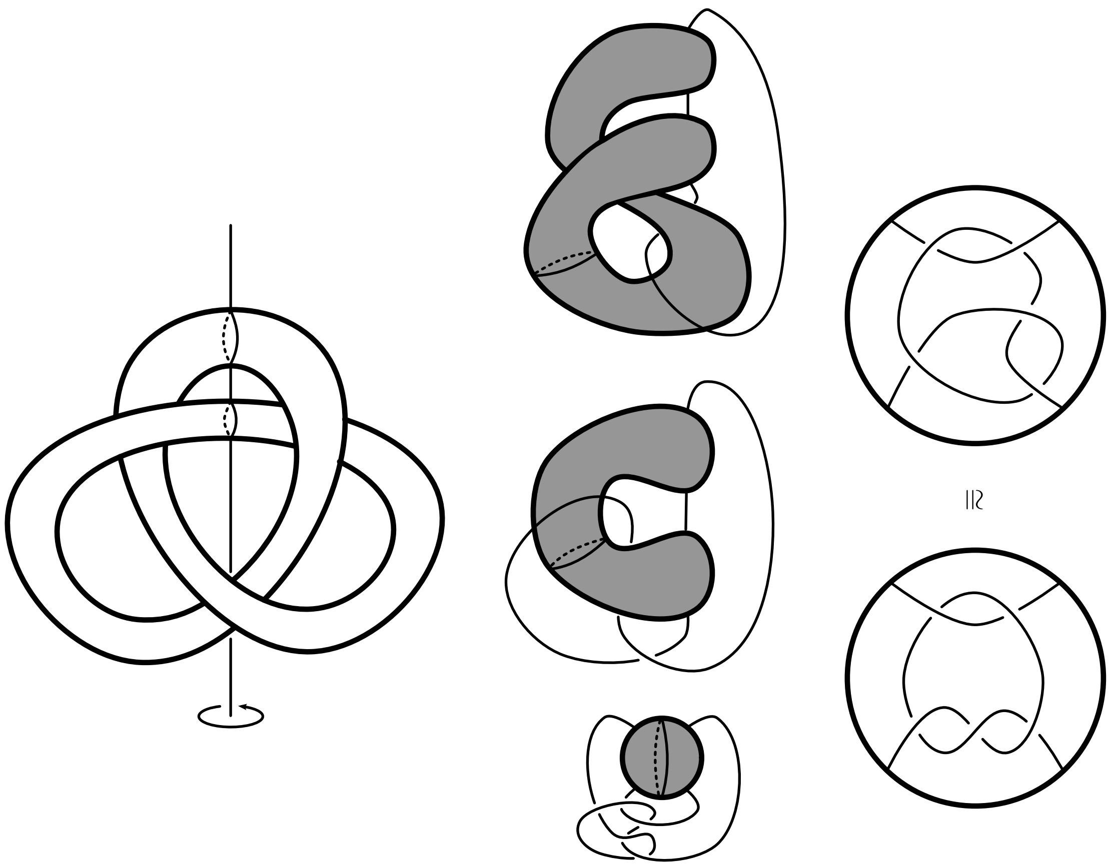


Figure 4.2 The trefoil with its strong inversion (left), an isotopy of a fundamental domain for the involution (centre), and two homeomorphic views of the tangle associated to the quotient (right). Notice that both representatives of the tangle have the property that $\tau(\frac{1}{0})$ is the trivial knot, giving a branch set for the trivial surgery.

complement of the trefoil is Seifert fibered over $D^2(2,3)$*. Indeed, the two tangles lift to a pair of solid tori identified along an essential annulus, the cores of which are singular fibres of order 2 and 3.*

There is a large class of examples from which to draw, since many "small" knots[3] turn out to be strongly invertible. The same is true for some familiar classes of knots: all two-bridge knots are strongly invertible (see (Montesinos, 1976)), as are all torus knots by

[3]Deliberately impresise, but we take this to mean knots with up to 11 crossings (i.e. those found in Rolfsen's table), say.

a result of Schreier (Schreier, 1924). More generally, Berge knots provide an interesting class of strongly invertible knots (Osborne, 1981), since they embed on a Heegaard surface of genus 2.

It is possible to work with a slightly larger class of manifolds with torus boundary. In general, a strong inversion on a manifold with torus boundary will refer to an involution with 1-dimensional fixed point set intersecting the boundary transversally in 4 points, as in Definition 1.11.

Definition 4.6. *Given an irreducible knot manifold M with $H_1(M;\mathbb{Q})=\mathbb{Q}$, suppose that there is a strong inversion $f\in\mathrm{End}(M)$ with the property that M/f is homeomorphic to a ball. Such M will be called a simple, strongly invertible knot manifold.*

For a given simple, strongly invertible knot manifold, there is always a tangle $T=(B^3,\tau)$ associated to the quotient of the strong inversion. Thus $M=\mathbf{\Sigma}(B^3,\tau)$, where $T=(B^3,\tau)$ will be referred to as the *associated quotient tangle*. Note that in the presence of *multiple* strong inversions, this tangle is not unique and depends on a fixed choice of involution. In the given notation, T refers to the homeomorphism class of the tangle, while τ will denote a given choice of representative. Such a choice will often arise as a choice of diagram for the tangle.

Notice that, by construction, there is a natural operation of reflection on a simple strongly invertible knot manifold given by $M^\star=\mathbf{\Sigma}(B^3,\tau^\star)$ where $M=\mathbf{\Sigma}(B^3,\tau)$ and $\tau^\star$ denotes the mirror of the branch set.

While complements of strongly invertible knots in S^3 provide the primary source of examples of simple, strongly invertible knot manifolds, we remark that the latter is certainly a much larger class. For example, the exterior of a generalized torus knot – those manifolds Seifert fibered over the disk with two cone points – always provides such a manifold. The following is due to Montesinos.

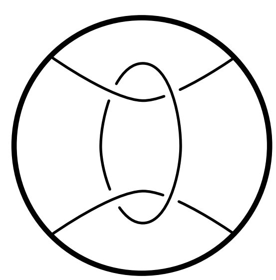
Proposition 4.7. *(Montesinos, 1976) Let Y be a Seifert fibre space with base orbifold $S^2(p,q,r)$. Then $Y\cong M(\alpha)$ where M is a simple strongly invertible knot manifold and*

M has Seifert fibre structure with base orbifold $D^2(p,q)$.

Proof. Let M be a knot manifold endowed with a Seifert fibre structure and suppose that the base orbifold is $D^2(p,q)$, the disk with two cone points. We may assume that $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$, and that the cone points p, q lie to either side of 0 on the real axis in the interior of D^2. Note that such a Seifert fibre space is a union of solid tori along an essential annulus that corresponds to the lift of the imaginary axis in the interior of D^2. As we have noted previously, the solid torus admits a strong inversion, and such a strong inversion fixes the singular fibre of any Seifert fibre structure on the solid torus. In particular, the solid torus as a Seifert fibre space has base orbifold D^2 with a single cone point, and the strong inversion corresponds to a reflection in the real axis. Now the reflection $\rho(z) = \bar{z}$ in the real axis (fixing the cone points p, q) lifts to a strong inversion on M, and ρ fixes the singular fibres.

Choose a regular fibre $\varphi \subset \partial M$. By Theorem 1.21, the Dehn filling $M(\varphi)$ must be a connect sum of lens spaces. Further, extending the strong inversion across the surgery torus gives a strong inversion on $M(\varphi)$, the quotient of which is S^3 (Montesinos, 1976). As a result, $M/f \cong B^3$ as in the proof of Proposition 4.4.

Now suppose that Y is Seifert fibered, with base orbifold $S^2(p,q,r)$. Removing a tubular neighbourhood of a singular fibre yields a knot manifold M that is Seifert fibered with base orbifold $D^2(p,q)$. Such an M must be simple and strongly invertible. □



As a particular example, it follows that the twisted I-bundle over the Klein bottle is a simple, strongly invertible knot manifold (this manifold is not the complement of a knot in S^3, but rather the complement of a knot representing twice the generator of the first homology in $S^2 \times S^1$). The associated quotient tangle for this manifold is shown on the right; note that this is the unique manifold with a $D^2(2,2)$ structure (see, for example, (Montesinos, 1976)) and this structure, arising as the identification of two fibered solid tori (each with base orbifold $D^2(2)$) along an essential annulus, is reflected in this tangle as the sum of rational tangles.

4.4 Branch sets for Dehn fillings

For a given simple strongly invertible knot manifold M, any representative of the associated quotient tangle T has a pair of distinguished arcs $(\gamma_{\frac{1}{0}}, \gamma_0)$ in the boundary as illustrated in Figure 4.3 that meet in a single point. The hemisphere (that is, the eastern and southern hemisphere) containing each arc lifts to an annulus in $\partial M = \mathbf{\Sigma}(\partial B^3, \partial\tau)$, so that the pair $(\gamma_{\frac{1}{0}}, \gamma_0)$ lifts to a (unoriented) basis for $H_1(\partial M; \mathbb{Z})$. By fixing an orientation so that $\widetilde{\gamma}_{\frac{1}{0}} \cdot \widetilde{\gamma}_0 = 1$, we obtain a basis for Dehn fillings of M.

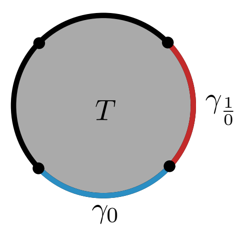


Figure 4.3 The arcs $\gamma_{\frac{1}{0}}$ (red) and γ_0 (blue) in the boundary of T.

Let $\frac{p}{q} = [a_1, \ldots, a_r]$ denote the continued fraction expansion

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots \frac{1}{a_r}}}$$

where $a_1 \geq 0$ and $a_i > 0$ for $i > 1$ when $\frac{p}{q} \geq 0$ (when $\frac{p}{q} \leq 0$, $a_1 \leq 0$ and $a_i < 0$ for $i > 1$). To $\frac{p}{q}$ we associate the braid

$$\beta = \begin{cases} \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_1^{-a_r} & r \text{ even} \\ \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_2^{a_r} & r \text{ odd} \end{cases}$$

Now observing that $0 = [0]$, and fixing the convention $\frac{1}{0} = [\,]$ (with *length* $r = 0$), denote by $\tau(\frac{p}{q})$ the link obtained by the closure of T^β depending on whether r is even or odd as in Figure 4.4 (a particular example is shown in Figure 4.5).

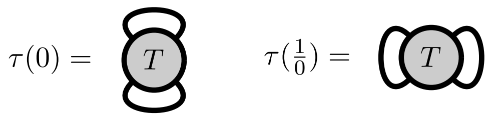


Figure 4.4 The odd-closure $\tau(0)$ and the even-closure $\tau(\frac{1}{0})$ of the tangle T.

Now the strong inversion on M extends to an involution on a Dehn filling of M, giving rise to a two-fold branched cover of S^3, branched over a link that we may now make explicit.

Proposition 4.8. *Let M be a simple strongly invertible knot manifold. For a given slope $\alpha = p\widetilde{\gamma}_{\frac{1}{0}} + q\widetilde{\gamma}_0$ we have that $\mathbf{\Sigma}(S^3, \tau(\frac{p}{q})) \cong M(\alpha)$.*

Sketch of proof. First observe that $\mathbf{\Sigma}(S^3, \tau(0)) \cong M(\widetilde{\gamma}_0)$ and $\mathbf{\Sigma}(S^3, \tau(\frac{1}{0})) \cong M(\widetilde{\gamma}_{\frac{1}{0}})$.

Now consider the action of σ_2. We claim that this half twist (viewed as an action on the disk with 2 marked points) lifts to a Dehn twist along the curve $\widetilde{\gamma}_{\frac{1}{0}}$. Indeed, the two fold branched cover of this disk is an essential annulus in ∂M (c.f (Rolfsen, 1976, Chapter 10)). In terms of the basis $(\widetilde{\gamma}_{\frac{1}{0}}, \widetilde{\gamma}_0)$, this Dehn twist may be written $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Similarly, the action of σ_1^{-1} lifts to a Dehn twist about $\widetilde{\gamma}_0$; this takes the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

In particular, we have that $\mathbf{\Sigma}(S^3, \tau(n)) \cong M(n\widetilde{\gamma}_{\frac{1}{0}} + \widetilde{\gamma}_0)$ and $\mathbf{\Sigma}(S^3, \tau(\frac{1}{n})) \cong M(\widetilde{\gamma}_{\frac{1}{0}} + n\widetilde{\gamma}_0)$.

In general, for $\frac{p}{q} = [a_1, \ldots, a_r]$, the action of the associated braid may be written (in the case r is even) as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_r} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{a_1}$$

(the case r odd differs only in the first matrix of this product). We leave it to the reader to check that the first column of the resulting matrix is $\begin{pmatrix} q \\ p \end{pmatrix}$ so that we have specified the filling slope $\alpha = p\widetilde{\gamma}_{\frac{1}{0}} + q\widetilde{\gamma}_0$ as desired. Details may be found in Rolfsen (Rolfsen, 1976, Chapter 10). $\square$

Corollary 4.9. *Given a basis (α, β) for surgery in ∂M there is a choice of representative for T so that $(\gamma_{\frac{1}{0}}, \gamma_0)$ lifts to (α, β).*

Proof. For any choice of representative of T, write $\alpha = p\widetilde{\gamma}'_{\frac{1}{0}} + q\widetilde{\gamma}'_0$. In terms of this representative then, $M(\alpha) = \mathbf{\Sigma}(S^3, \tau(\frac{p}{q}))$. However, by removing the arcs forming the closure as in Figure 4.4, the resulting tangle may be viewed as a reframing of T, and yields a representative compatible with α. By twisting, this representative may be made compatible with (α, β) since $\alpha \cdot \beta = 1$. □

As a result, for any choice of basis (α, β) for Dehn surgery on a simple strongly invertible knot manifold, a *compatible* representative for the associated quotient tangle exists so that $\alpha = \widetilde{\gamma}_{\frac{1}{0}}$ and $\beta = \widetilde{\gamma}_0$. Notice that, as a result of Lemma 1.5, we have that

$$\det(\tau(\tfrac{p}{q})) = c_M \Delta(p\widetilde{\gamma}_{\frac{1}{0}} + q\widetilde{\gamma}_0, \lambda_M) = c_M \Delta(p\alpha + q\beta, \lambda_M)$$

once a basis for Dehn surgery, and compatible associated quotient tangle have been fixed. In particular, given a strongly invertible knot in S^3 there is always a choice of associated quotient tangle for which

$$S^3_{p/q}(K) = \mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q})).$$

Such a representative will be referred to as the canonical representative for the associated quotient tangle.

The fact that Dehn surgery on simple, strongly invertible knot manifolds may be viewed as a rational tangle attachment in the branch set is a generalization – or perhaps, incarnation – of the Montesinos trick (Montesinos, 1975), which says that an unknotting number one knot has two-fold branch cover that may be obtained by half-integer surgery on some other knot in S^3. More generally, we have:

Proposition 4.10. *The two-fold branched cover of a tangle unknotting number one knot may be obtained by surgery on a knot in S^3.*

Proof. Let $K = T_0 \cup T_1$ be a tangle unknotting number one knot (as in Definition 4.1), and $U = T_0 \cup T_2$ the corresponding tangle decomposition of the trivial knot. Since T_1 and T_2 are rational, the branched covers $\mathbf{\Sigma}(S^3, K)$, $\mathbf{\Sigma}(S^3, U)$ differ only in a solid torus, that is, by a Dehn surgery. On the other hand, the branched cover of the trivial knot is the three-sphere. □

Finally, the arguments in the sequel simplify considerably due to the following observation which allows us, up to mirrors, to consider only positive surgery coefficients (and hence restrict to positive continued fractions).

Proposition 4.11. *Let M be a simple strongly invertible knot manifold, together with a fixed basis for Dehn surgery and compatible associated quotient tangle. Then*

$$\mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q})) \cong -\mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q})^\star) \cong -\mathbf{\Sigma}(S^3, \tau^\star(-\tfrac{p}{q})).$$

Proof. Since $M^\star = \mathbf{\Sigma}(B^3, \tau^\star)$, we have that $M^\star(\alpha) = \mathbf{\Sigma}(S^3, \tau(\frac{p}{q})^\star)$. From the definition for $\tau(\frac{p}{q})$, it follows that $\tau(\frac{p}{q})^\star \simeq \tau^\star(-\frac{p}{q})$. □

4.5 On continued fractions

There are three fundamental properties for continued fractions relating to Dehn filling that will be essential for the inductive arguments that follow. Since it will always be possible as a result of Proposition 4.11 to work with positive[4] surgeries – and hence positive continued fractions – by passing to the mirror image, we will state these properties for positive continued fractions only.

Therefore, we assume that $\frac{p}{q} = [a_1, \ldots, a_r]$ is positive, with $a_1 \geq 0$ and $a_i > 0$ for all $i > 1$

Property 4.12. $\lfloor \frac{p}{q} \rfloor = a_1$ *and* $\lceil \frac{p}{q} \rceil = a_1 + 1$.

[4] More precisely, positive with respect to the rational longitude λ_M in the sense that the filling slope α has the property $\alpha \cdot \lambda_M > 0$.

Proof. It is immediate from the definition of $\frac{p}{q}$ as a continued fraction that $a_1 \leq \frac{p}{q} < a_1 + 1$ for $\frac{p}{q} = [a_1, \ldots, a_r]$. □

Property 4.13. $[a_1, \ldots, a_r, 1] = [a_1, \ldots, a_r + 1]$.

Proof. This is immediate from the partial evaluation of the continued fraction:

$$[a_1, \ldots, a_r, 1] = [a_1, \ldots, a_r + \tfrac{1}{1}] = [a_1, \ldots, a_r + 1]$$

□

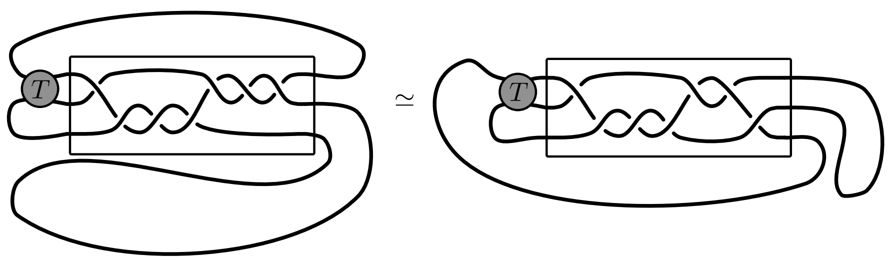


Figure 4.5 The link $\tau(\frac{13}{10})$ obtained from the odd-closure with the fraction $[1, 3, 3]$ (left), is isotopic to the link obtained from the even-closure with the fraction $[1, 3, 2 + 1] = [1, 3, 2, 1]$ (right).

It is important to note that this equality of continued fractions manifests itself as isotopic links when forming $\tau(\frac{p}{q})$, for any tangle. This results from the fact that the even- and odd-closures replace one another, as is illustrated in a particular case in Figure 4.5.

Finally, we turn to the behaviour of $\tau(\frac{p}{q})$ under resolutions.

Definition 4.14. *The terminal crossing of $\tau(\frac{p}{q})$ is the last crossing added by the action of $\beta \in B_3$ specified by the continued fraction. That is, the terminal crossing corresponds to the last generator in the braid word $\beta = \sigma_2^{a_1} \cdots \sigma_\epsilon^{a_r}$ (where σ_ϵ is either σ_2 or σ_1^{-1}, depending on the parity of r).*

Our convention will be that the terminal crossing of $\tau(\frac{p}{q})$ is resolved to obtain $\tau(\frac{p_0}{q_0})$ and $\tau(\frac{p_1}{q_1})$.

Property 4.15. $\frac{p}{q} = \frac{p_0+p_1}{q_0+q_1}$ *where* $\frac{p_0}{q_0} = [a_1, \ldots, a_{r-1}]$ *and* $\frac{p_1}{q_1} = [a_1, \ldots, a_{r-1}, a_r - 1]$.

Proof. Recall that a continued fraction may be recursively defined by convergents $\frac{h_n}{k_n}$ where $h_{-1} = 0, h_0 = 1$ and $h_n = a_n h_{n-1} + h_{n-2}$ for $n > 1$, and $k_{-1} = 1, k_0 = 0$ and $k_n = a_n k_{n-1} + k_{n-2}$ for $n > 1$.

Now write $\frac{h_{r-1}}{k_{r-1}} = \frac{p_0}{q_0}$ and $\frac{h_r}{k_r} = \frac{p_1}{q_1}$, then $\frac{p_0+p_1}{q_0+q_1} = \frac{h_r+h_{r-1}}{k_r+k_{r-1}} = [a_1, \ldots, a_r - 1, 1]$, so that applying Property 4.13 we have $\frac{p_0+p_1}{q_0+q_1} = [a_1, \ldots, a_r] = \frac{p}{q}$ as claimed. □

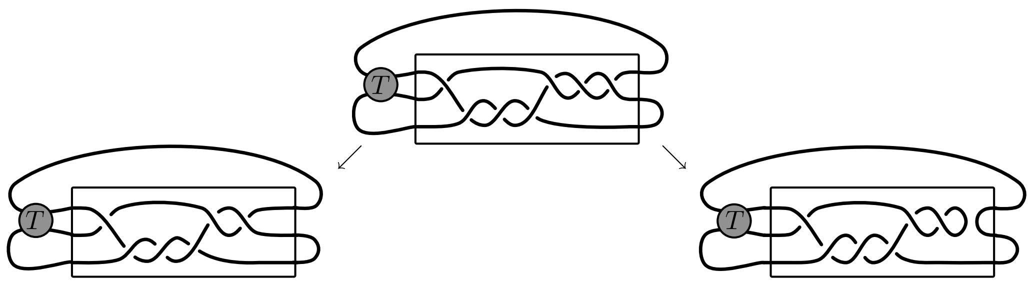


Figure 4.6 Resolving the terminal crossing of $\tau(\frac{13}{10}) = \tau[1,3,3]$ gives 0-resolution with $\frac{p_1}{q_1} = [1,3,2] = \frac{9}{7}$ and one resolution with $\frac{p_0}{q_0} = [1,3] = \frac{4}{3}$.

A particular example of Property 4.15 is illustrated in Figure 4.6. Notice that $\frac{p_0}{q_0}$ corresponds to the 0-resolution when r is even, and the 1-resolution otherwise. Similarly, $\frac{p_1}{q_1}$ corresponds to the 1-resolution when r is even, and the 0-resolution otherwise.

When $\frac{p}{q} = [a_1, \ldots, a_r]$ we will use the notation $\tau(\frac{p}{q}) = \tau[a_1, \ldots, a_r]$ for the closure when convenient.

4.6 Triads for tangles

Suppose α and β are a pair of slopes in ∂M with $\alpha \cdot \beta = +1$. Fix a compatible representative for the associated quotient tangle $T = (B^3, \tau)$ with the property that $M(\alpha) = \mathbf{\Sigma}(S^3, \tau(\frac{1}{0}))$ and $M(\beta) = \mathbf{\Sigma}(S^3, \tau(0))$.

Proposition 4.16. *If* $\tau(\frac{1}{0})$ *and* $\tau(0)$ *are quasi-alternating, and* $\alpha \cdot \lambda_M, \beta \cdot \lambda_M > 0$*, then* $\tau(1)$ *is quasi-alternating as well.*

Remark 4.17. *Note that the quasi-alternating hypothesis ensures that neither* α *nor* β *coincides with the rational longitude.*

Proof of Proposition 4.16. We need to calculate $\det(\tau(1))$. To this end, by applying Lemma 1.5 we have that

$$\begin{aligned}\det(\tau(1)) &= |H_1(M(\alpha+\beta);\mathbb{Z})| \\ &= c_M\Delta(\alpha+\beta,\lambda_M) \\ &= c_M|(\alpha+\beta)\cdot\lambda_M| \\ &= c_M|\alpha\cdot\lambda_M+\beta\cdot\lambda_M| \\ &= c_M|\alpha\cdot\lambda_M| + c_M|\beta\cdot\lambda_M| \\ &= c_M\Delta(\alpha,\lambda_M)+c_M\Delta(\beta,\lambda_M) \\ &= |H_1(M(\alpha);\mathbb{Z})| + |H_1(M(\beta);\mathbb{Z})| \\ &= \det(\tau(\tfrac{1}{0}))+\det(\tau(0)),\end{aligned}$$

which verifies that $\tau(1)$ is a quasi-alternating link, since both $\tau(\frac{1}{0})$ and $\tau(0)$ are quasi-alternating. □

Remark 4.18. *Notice that the condition on intersection with* λ_M *may be relaxed at the expense of taking mirrors. For any* $M(\alpha)$ *and* $M(\beta)$ *with quasi-alternating branch sets* $\tau(\frac{1}{0})$ *and* $\tau(0)$ *we can ensure positive intersection with* λ_M *at the expense of* $\alpha\cdot\beta=\pm1$. *In the case that* $\alpha\cdot\beta=-1$, *the same argument works by passing to mirrors. Any quasi-alternating link has quasi-alternating mirror image so that if* $\tau(\frac{1}{0})$ *and* $\tau(0)$ *are quasi-alternating then one of* $\tau(-1)$ *or* $\tau(1)$ *is quasi-alternating.*

Definition 4.19. *A triad of links* $(\tau(\frac{1}{0}),\tau(0),\tau(1))$ *corresponds to a triple* $(\alpha,\beta,\alpha+\beta)$ *where* $\alpha\cdot\beta=1$, $\alpha\cdot\lambda_M>0$, *and* $\beta\cdot\lambda_M>0$.

The requirement that α and β intersect positively with λ_M is stronger than necessary, since it is attainable up to taking mirrors. However, with this assumption we have:

Theorem 4.20. *A triad of links, for which* $\tau(\frac{1}{0})$ *and* $\tau(0)$ *are quasi-alternating, gives rise to an infinite family of links* $\tau(\frac{p}{q}) \in \mathcal{Q}$*, for* $\frac{p}{q} \geq 0$.

Proof. First observe that $\tau(n)$ is quasi-alternating for every $n \geq 0$. This is immediate by induction on n, since

$$\begin{aligned}
\det(\tau(n)) &= |H_1(M(n\alpha+\beta);\mathbb{Z})| \\
&= c_M\Delta(n\alpha+\beta,\lambda_M) \\
&= c_M|(n\alpha+\beta)\cdot\lambda_M| \\
&= c_M|n\alpha\cdot\lambda_M+\beta\cdot\lambda_M| \\
&= c_M|\alpha\cdot\lambda_M| + c_M|((n-1)\alpha+\beta)\cdot\lambda_M| \\
&= c_M\Delta(\alpha,\lambda_M) + c_M\Delta((n-1)\alpha+\beta,\lambda_M) \\
&= |H_1(M(\alpha);\mathbb{Z})| + |H_1(M((n-1)\alpha+\beta);\mathbb{Z})| \\
&= \det(\tau(\tfrac{1}{0})) + \det(\tau(n-1)),
\end{aligned}$$

with Proposition 4.16 providing a base case.

For $\tau(\frac{p}{q})$, we need a second induction in the length of the continued fraction $\frac{p}{q} = [a_1,\ldots,a_r]$. The base case $r=1$ is the observation above that $\tau(n)$ is quasi-alternating, applying Property 4.12.

Suppose then that $\tau(\frac{p}{q})$ is quasi-alternating for all $\frac{p}{q} \geq 0$ that may be represented by a continued fraction of length $r-1$. By resolving the terminal crossing and applying

Property 4.15,

$$\begin{aligned}
\det(\tau(\tfrac{p}{q})) &= |H_1(M(p\alpha + q\beta); \mathbb{Z})| \\
&= c_M \Delta(p\alpha + q\beta, \lambda_M) \\
&= c_M |(p\alpha + q\beta) \cdot \lambda_M| \\
&= c_M |(p_0 + p_1)\alpha \cdot \lambda_M + (q_0 + q_1)\beta \cdot \lambda_M| \\
&= c_M |(p_0\alpha + q_0\beta) \cdot \lambda_M| + c_M |(p_1\alpha + q_1\beta) \cdot \lambda_M| \\
&= c_M \Delta(p_0\alpha + q_0\beta, \lambda_M) + c_M \Delta(p_1\alpha + q_1\beta, \lambda_M) \\
&= |H_1(M(p_0\alpha + q_0\beta); \mathbb{Z})| + |H_1(M(p_1\alpha + q_1\beta); \mathbb{Z})| \\
&= \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))
\end{aligned}$$

where $\frac{p_0}{q_0} = [a_1, \ldots, a_r - 1]$ and $\frac{p_1}{q_1} = [a_1, \ldots, a_{r-1}]$ when r is odd (and these are switched when r is even). In either case, we are reduced to a continued fraction of length $r - 1$ which must be quasi-alternating by our induction hypothesis, and a continued fraction $[a_1, \ldots, a_r - 1]$ with r^{th} entry reduced by one.

Since $[a_1, \ldots, a_{r-1}, 1] = [a_1, \ldots, a_{r-1} + 1]$ by Property 4.13, repeating the above argument $a_r - 1$ times completes the induction. □

Remark 4.21. *We point out that Theorem 4.20 can be very useful if one is interested in constructing infinite families of quasi-alternating links. Indeed, this has been pursued in (Champanerkar and Kofman, 2007; Widmer, 2008) to construct such families of Montesinos links. However, these examples are verified using combinatorial methods for computing the determinant. By using the associated Dehn filling to control the determinant, a wider range of examples seem accessible. Indeed, it seems likely that the examples in both works may be recovered via Theorem 4.20.*

4.7 Branch sets for L-spaces obtained from Berge knots

Theorem 4.20 gives a tool with which to study the overlap between the various classes of L-spaces introduced in Chapter 3.

Proposition 4.22. *For large enough integer surgery coefficient N, the branch set for $S^3_N(K)$ is quasi-alternating whenever K is a Berge knot (up to possibly replacing K by its mirror). Moreover, for every $\frac{p}{q} > N$ the branch set associated to $\frac{p}{q}$-surgery on K must be quasi-alternating.*

Proof. For any Berge knot K there is some integer N, positive up to taking mirrors, with the property that $S^3_N(K)$ is a lens space. As a result, $(\mu, N\mu+\lambda, (N+1)\mu+\lambda)$ gives a triad of slopes, in terms of the canonical basis (μ,λ) for K.

Moreover, since Berge knots are strongly invertible, there is an associated quotient tangle $T = (B^3, \tau')$ with representative so chosen so that $\tau'(\frac{1}{0})$ is unknotted, and $S^3_N(K) = \mathbf{\Sigma}(S^3, \tau'(0))$. By construction, both branch sets are quasi-alternating: the trivial knot $\tau'(\frac{1}{0})$ and some non-split 2-bridge link $\tau'(0)$.

Now applying Theorem 4.20, $\tau'(\frac{p}{q})$ must be quasi-alternating for every $\frac{p}{q} \geq 0$, so that the L-space $S^3_{(Nq+p)/q}(K)$ is branched over S^3 with quasi-alternating branch set $\tau'(\frac{p}{q})$.[5] $\square$

As a result, many[6] of the L-spaces arising as surgery on a Berge knot are also obtained as two-fold branched covers of quasi-alternating links. This implies in particular that the corresponding branch sets have thin Khovanov homology. Although this cannot be the case for all possible fillings when K is non-trivial (c.f. Proposition 3.35), it turns out that in terms of homological width, the branch set corresponding to a filling of a Berge knot cannot be too much more complicated.

Proposition 4.23. *Surgery on a Berge knot has branch set with width at most 2.*

Proof. Fix a representative $T = (B^3, \tau')$ for the associated quotient tangle of K that is compatible with the basis for surgery $(\mu, N\mu+\lambda)$, as in Proposition 4.22. According to

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[5]Notice that if $N \leq \frac{r}{q}$ then $Nq \leq r$ so that $r = Nq+p$ for $p \geq 0$ and $\frac{r}{q} = \frac{Nq+p}{q}$.

[6]Though not all: consider the Poincaré homology sphere, for example.

Proposition 4.22 then, $\tau'(\frac{p}{q})$ is homologically thin for all $\frac{p}{q} > N$ (up to taking mirrors) by virtue of being quasi-alternating.

Notice in particular that $\tau'(1)$ is a homologically thin link, with

$$\widetilde{\mathrm{Kh}}(\tau'(1)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau'(0))[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau'(\tfrac{1}{0}))[-\tfrac{c}{2}] \right) \tag{4.1}$$

where $c = n_-(\tau'(\frac{1}{0})) - n_-(\tau'(1)) = n_-(\tau'(\frac{1}{0})) - n_-(\tau'(0))$. That the resolved crossing of $\tau'(1)$ is positive follows immediately from the fact that $\tau'(\frac{1}{0})$ is a trivial knot. Recall that our choice of orientation is arbitrary, though fixed, since we are working with Khovanov homology as a relatively $\mathbb{Z}$-graded group in this setting (c.f. Section 2.4).

Similarly,

$$\widetilde{\mathrm{Kh}}(\tau'(-1)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau'(\tfrac{1}{0}))[-\tfrac{c'+1}{2}] \to \widetilde{\mathrm{Kh}}(\tau'(0))[\tfrac{1}{2}] \right)$$

where $c' = n_-(\tau'(\frac{1}{0})) - n_-(\tau'(-1)) = n_-(\tau'(\frac{1}{0})) - n_-(\tau'(0)) - 1 = c - 1$ so that

$$\begin{aligned} \widetilde{\mathrm{Kh}}(\tau'(-1)) &\cong H_* \left(\widetilde{\mathrm{Kh}}(\tau'(\tfrac{1}{0}))[-\tfrac{c}{2}] \to \widetilde{\mathrm{Kh}}(\tau'(0))[-\tfrac{1}{2}][1] \right) \\ &\cong H_* \left(\widetilde{\mathrm{Kh}}(\tau'(\tfrac{1}{0}))[-\tfrac{c}{2}][-1] \to \widetilde{\mathrm{Kh}}(\tau'(0))[-\tfrac{1}{2}] \right)[1] \end{aligned} \tag{4.2}$$

Now by ignoring the overall shift of [1] since we are working with the relatively δ-graded group, $w(\tau'(1)) = 1$ implies that $w(\tau'(-1)) \leq 2$ (this follows from comparison of the expressions (4.1) and (4.2)).

In general, if $n > 0$ then

$$\widetilde{\mathrm{Kh}}(\tau'(-n)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau'(\tfrac{1}{0}))[-\tfrac{c-n+1}{2}] \to \widetilde{\mathrm{Kh}}(\tau'(-n+1))[\tfrac{1}{2}] \right)$$

so that by iteratively applying this sequence we obtain

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau'(-n)) &\cong H_*\left(\bigoplus_n \widetilde{\mathrm{Kh}}(\tau'(\tfrac{1}{0}))[-\tfrac{c-n+1}{2}] \to \widetilde{\mathrm{Kh}}(\tau'(0))[\tfrac{n}{2}]\right)\\ &\cong H_*\left(\bigoplus_n \widetilde{\mathrm{Kh}}(\tau'(\tfrac{1}{0}))[-\tfrac{c}{2}][-1] \to \widetilde{\mathrm{Kh}}(\tau'(0))[-\tfrac{1}{2}]\right)[\tfrac{n+1}{2}] \qquad (4.3)\end{aligned}$$

As a result, $\tau'(-n)$ may be computed for all $n > 0$ in terms of $\tau'(0)$ and $\tau'(\frac{1}{0})$, and it follows that $w(\tau'(-n)) \leq 2$ for all $n > 0$. Note that width 2 must occur: $\det(L) = 0$ implies $w(L) > 1$ according to Proposition 2.12. Nevertheless, we obtain the bound as claimed for all branch sets associated to integer fillings: $w(\tau'(n)) \leq 2$ for every integer n.

The key observation at this stage is that

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}(\tau'(n+1))[x]\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}(\tau'(n))[x']\right)$$

as absolutely $\mathbb{Z}$-graded groups, when shifted (by some $[x]$ and $[x']$) according to the mapping cones above (compare Equations (4.1), (4.2) and (4.3)). In particular, it follows that

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau'(n+1))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau'(n))\right)$$

since $\det(\tau'(n+1)) = \det(\tau'(n)) + 1$ (and applying Proposition 2.14 or Proposition 2.15).

To conclude the proof, fix the canonical representative for the associated quotient tangle. That is, $S^3_0(K) = \mathbf{\Sigma}(S^3, \tau(0))$ while $\tau(\frac{1}{0})$ is the trivial knot as before. We will show that

$$w(\tau(\tfrac{p}{q})) \leq \max\left\{w(\tau\lfloor\tfrac{p}{q}\rfloor), w(\tau\lceil\tfrac{p}{q}\rceil)\right\} \leq 2$$

for every $\frac{p}{q}$-surgery.

As before, we need only consider $\frac{p}{q} > 0$; the case $\frac{p}{q} < 0$ follows by considering mirrors. Since $w(\tau(n)) \leq 2$, we have a base case for induction in the length of continued fraction

$\frac{p}{q} = [a_1, \ldots, a_r]$. Suppose then that the result holds for all continued fractions of length $r-1$, and that either

$$\text{Supp}\left(\widetilde{\text{Kh}}_\sigma(\tau[a_1, \ldots, a_r - 1])\right) \subseteq \text{Supp}\left(\widetilde{\text{Kh}}_\sigma(\tau[a_1, \ldots, a_{r-1}])\right)$$

or the inclusion is reversed (recall that the 0- and 1-resolutions alternate roles, depending on parity). We proceed in two cases.

Case 1: $1 < \frac{p}{q}$

By resolving the the terminal crossing of $\tau(\frac{p}{q}) = \tau[a_1, \ldots, a_r]$ we have that

$$\begin{aligned}
\det(\tau(\tfrac{p}{q})) &= |H_1(M(p\mu + q\lambda); \mathbb{Z})| \\
&= |H_1(S^3_{p/q}(K); \mathbb{Z})| \\
&= p \\
&= p_0 + p_1 \\
&= |H_1(S^3_{p_0/q_0}(K); \mathbb{Z})| + |H_1(S^3_{p_1/q_1}(K); \mathbb{Z})| \\
&= \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))
\end{aligned}$$

since we have fixed a representative compatible with the canonical framing. Notice that $1 \leq \frac{p_0}{q_0}, \frac{p_1}{q_1}$, so that we are in a position to apply Proposition 2.14:

$$\begin{aligned}
&\widetilde{\text{Kh}}_\sigma(\tau[a_1, \ldots, a_r]) \\
&\cong \begin{cases} H_*\left(\widetilde{\text{Kh}}_\sigma(\tau[a_1, \ldots, a_r - 1]) \to \widetilde{\text{Kh}}_\sigma(\tau[a_1, \ldots, a_{r-1}])\right) & \text{for } r \text{ odd} \\ H_*\left(\widetilde{\text{Kh}}_\sigma(\tau[a_1, \ldots, a_{r-1}]) \to \widetilde{\text{Kh}}_\sigma(\tau[a_1, \ldots, a_r - 1])\right) & \text{for } r \text{ even} \end{cases}
\end{aligned}$$

By iterative application of Proposition 2.14 then we have

$$\begin{aligned}&\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,\ldots,a_r])\\ &\cong \begin{cases} H_*\left(\bigoplus_{a_r-1}\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,\ldots,a_{r-1}+1])\to\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,\ldots,a_{r-1}])\right) & \text{for } r \text{ odd}\\ H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,\ldots,a_{r-1}])\to\bigoplus_{a_r-1}\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,\ldots,a_{r-1}+1])\right) & \text{for } r \text{ even}\end{cases}\end{aligned}$$

Now the result follows from the inductive hypothesis. Notice that the expression above is an abuse of notation: there may be further differentials to consider among the groups of $\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,\ldots,a_{r-1}+1])$, however this can only lower the width and can be safely ignored in the present setting. Therefore for $1\geq\frac{p}{q}$ (indeed, $1\geq|\frac{p}{q}|$ after considering mirrors) we have that $w(\tau(\frac{p}{q}))\leq 2$.

Case 2: $0<\frac{p}{q}<1$

The argument is this case is identical, except when passing from length 2 to length 1: this step relies on the degenerative version of Manolescu and Ozsváth's exact sequence in Proposition 2.15.

When $r=2$ we have $\frac{p}{q}=[0,a_2]$ so that

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[0,a_2])\cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(0))[\tfrac{1}{2}]\to\widetilde{\mathrm{Kh}}_\sigma(\tau[0,a_2-1])\right)$$

since $\det(\tau(0))=0$ and $\det(\tau[0,a_2-1])=\det(\tau[0,a_2])$, hence

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[0,a_2])\cong H_*\left(\bigoplus_{a_2-1}\widetilde{\mathrm{Kh}}_\sigma(\tau(0))[\tfrac{1}{2}]\to\widetilde{\mathrm{Kh}}_\sigma(\tau[0,1])\right)$$

bearing in mind that there are possible differentials among the $\widetilde{\mathrm{Kh}}_\sigma(\tau(0))$. However, notice that $\widetilde{\mathrm{Kh}}_\sigma(\tau[0,1])=\widetilde{\mathrm{Kh}}_\sigma(\tau(1))$ and this may be written as

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(1))\cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(0))[\tfrac{1}{2}]\to\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{1}{0}))\right)$$

which we know to be of width at most 2. Indeed, in showing that this was the case (compare (4.2) and (4.3) bearing in mind the framing change of $0 \mapsto -n$), we might have observed that the group $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))$ is "added" to the second diagonal that must be present in $\widetilde{\mathrm{Kh}}(\tau(0))$ (and may or may not be present in $\widetilde{\mathrm{Kh}}(\tau(1))$). This is precisely the requirement on the support of these groups in the previous case, giving rise to the inductive hypothesis, and is enough to force $w(\tau[0,a_2]) \leq 2$, completing the proof. □

4.8 Manifolds with finite fundamental group

Combining work of Hodgson and Rubinstein with work of Lee, we have the following statement:

Theorem 4.24. *If Y is a lens space, then Y is a two fold branched cover of S^3, with branch set of width 1.*

Proof. Hodgeson and Rubinstein show that Y is a lens space if and only if it is the two-fold branched cover of a non-split two-bridge link (Hodgson and Rubinstein, 1985) (see Theorem 1.13); this family of links is alternating, hence thin, by Lee's result (Lee, 2005) (see also Section 5.7). □

Note that this excludes the manifold $S^2 \times S^1$ since it is branched over the 2-component trivial link having width 2.

Our main goal is to prove an analogous statement in the case of manifolds with finite fundamental group.

Theorem 4.25. *(Watson, 2008b) A manifold with finite fundamental group is a two-fold branched cover of S^3, with branch set of width at most 2.*

Proof. Manifolds with finite fundamental group are all Seifert fibered, and are either lens spaces or Seifert fibered over S^2 with 3 singular fibres (see Propostion 1.16, as well as Remark 1.17 and Remark 4.26). Due to Theorem 4.24, we need only consider the latter.

According to Proposition 1.16, there are two families of base orbifolds to consider: either $S^2(2,3,n)$ for $n = 3,4,5$, or $S^2(2,2,n)$ for $n > 1$.

In each case, the manifolds in question may be constructed by filling Seifert fibered manifolds (with boundary) with base orbifold $D^2(2,3)$ and $D^2(2,2)$, respectively. Note that the trefoil exterior and the twisted I-bundle over the Klein bottle are the unique Seifert fibered manifolds with base orbifolds $D^2(2,3)$ and $D^2(2,2)$, respectively.

In light of Theorem 1.14, it is enough to consider the branch sets related to these fillings, since each of the manifolds in question branches over S^3 in a unique way. Each of the resulting branch sets – which exist by virtue of Proposition 4.7 – is a Montesinos link composed of 3 rational tangles, encoding the Seifert structure in the cover (Montesinos, 1976).

In the first case, when filling the complement of the trefoil we appeal to Proposition 4.23: the branch set associated to filling a torus knot in S^3 has width at most 2.

To complete the proof then, we are left to consider the case of filling the twisted I-bundle over the Klein bottle, M.

When considered with Seifert structure having base orbifold $D^2(2,2)$, this manifold has the property that $\Delta(\varphi, \lambda_M) = 1$, where φ is a regular fibre in the boundary. Note that $M(\lambda_M)$ is $S^2 \times S^1$, and $M(n\varphi + \lambda_M)$ is a lens space for all $n \neq 0$ by applying Theorem 1.21. By fixing a representative for the associated quotient tangle compatible with the basis for surgery (φ, λ_M) it follows that $w(\tau(n)) = 1$ for all $n \neq 0$, and $w(\tau(0)) = 2$.

Now resolving the terminal crossing in $\tau(\frac{p}{q})$ we have, for $\frac{p}{q} \geq 0$,

$$\begin{aligned}
\det(\tau(\tfrac{p}{q})) &= |H_1(M(p\varphi + q\lambda_M);\mathbb{Z})| \\
&= c_M \Delta(p\varphi + q\lambda_M, \lambda_M) \\
&= c_M |p\varphi \cdot \lambda_M| \\
&= c_M |(p_0 + p_1)\varphi \cdot \lambda_M + (q_0 + q_1)\lambda_M \cdot \lambda_M| \\
&= c_M |(p_0\varphi + q_0\lambda_M) \cdot \lambda_M| + c_M |(p_1\varphi + q_1\lambda_M) \cdot \lambda_M| \\
&= c_M \Delta(p_0\varphi + q_0\lambda_M, \lambda_M) + c_M \Delta(p_1\varphi + q_1\lambda_M, \lambda_M) \\
&= |H_1(M(p_0\varphi + q_0\lambda_M);\mathbb{Z})| + |H_1(M(p_1\varphi + q_1\lambda_M);\mathbb{Z})| \\
&= \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))
\end{aligned}$$

in terms of (φ, λ_M).

This is enough to obtain the result, proceeding by double induction exactly as in the proof of Proposition 4.23, working with the basis (φ, λ_M) in place of the canonical basis (μ, λ). □

Remark 4.26. *The use of geometrization in this proof may be avoided by instead proving the statement that* **two-fold branched covers with finite fundamental group have branch set with width at most 2**. *Since our applications will always pertain to manifolds admitting an involution having non-empty fixed point set, we are naturally in the setting of the orbifold theorem, as discussed in Remark 1.17.*

Alternatively – in the same spirit as Proposition 3.18 – we have the following statement (that does not depend on geometrization in any form):

Theorem 4.27. *A manifold with elliptic geometry is a two-fold branched cover of S^3, with branch set of width at most 2.*

CHAPTER V

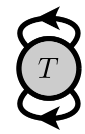
WIDTH BOUNDS FOR BRANCH SETS

In this chapter we turn our attention to the behaviour of the Khovanov homology of branch sets associated to surgery on simple, strongly invertible knot manifolds. For simplicity, we focus on the case of surgery on strongly invertible knots in S^3, though similar results may be obtained more generally (see Appendix, for example).

This material contained here is new, building on results in (Watson, 2008b), though width in Khovanov homology has received some attention recently. We point in particular to recent work of Lowrence studying the homological width of closed 3-braids (Lowrance, 2009). In this setting, though the focus of each paper is quite different, some of the results have a similar flavour.

5.1 A mapping cone for integer surgeries

Given a strongly invertible knot $K \hookrightarrow S^3$, with fixed strong inversion, let $T = (B^3, \tau)$ be the associated quotient tangle, compatible with the canonical framing (μ, λ). As in Chapter 4, we will refer to this as the canonical representative for the associated quotient tangle. Therefore, $\tau(\frac{1}{0})$ is the trivial knot, and $S^3_0(K) \cong \mathbf{\Sigma}(S^3, \tau(0))$. As a result, $\widetilde{\text{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, and $w(\tau(0)) > 1$ since $\det(\tau(0)) = 0$. Notice that $\tau(0)$ is a two component link.



In the interest of studying the Khovanov homology of the branch sets associated to integer surgery, we choose the orientation on $\tau(0)$ shown on the right. That this is possible follows from the fact that $\tau(\frac{1}{0})$ is the trivial knot; that such a choice is copacetic results from the fact that $\widetilde{\mathrm{Kh}}(\tau(0))$, in the present context, is a relatively bi-graded group.[1] With this orientation on $\tau(0)$, there is a natural constant related to a fixed diagram for the compatible representative of the associated quotient tangle

$$c_\tau = n_-(\tau(\tfrac{1}{0})) - n_-(\tau(0)).$$

Since $\tau(\frac{1}{0})$ has a single component, c_τ is independent of choice of orientation on $\tau(\frac{1}{0})$.

For example, we may rewrite the mapping cones in Khovanov homology as

$$\widetilde{\mathrm{Kh}}(\tau(1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{c_\tau}{2},\tfrac{3c_\tau+2}{2}]\right)$$

since $c = n_-(\tau(\frac{1}{0})) - n_-(\tau(1)) = n_-(\tau(\frac{1}{0})) - n_-(\tau(0)) = c_\tau$, and

$$\widetilde{\mathrm{Kh}}(\tau(-1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{c_\tau}{2},\tfrac{3c_\tau-2}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[\tfrac{1}{2},-\tfrac{1}{2}]\right)$$

since $c = n_-(\tau(\frac{1}{0})) - n_-(\tau(-1)) = n_-(\tau(\frac{1}{0})) - n_-(\tau(0)) - 1 = c_\tau - 1$. Notice that in this case there is an overall $[1,0]$ shift (which may be ignored, as our interest is in the relative gradings and not the absolute gradings) so that

$$\widetilde{\mathrm{Kh}}(\tau(-1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{c_\tau}{2},\tfrac{3c_\tau-2}{2}][-1,0] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{1}{2},-\tfrac{1}{2}]\right)[1,0]$$

which allows for comparison of the homology of $\tau(\pm 1)$ in terms of $\widetilde{\mathrm{Kh}}(\tau(0))$ and the new generator $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$.

More generally, we have:

---

[1] Only the absolute grading depends on orientation, as per Section 2.4, so we are free to fix any orientation for convenience so long as we remain consistent when computing using the skein exact sequence.

Lemma 5.1. *For any integer m, and positive integer n,*

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \bigoplus_n \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))\right)$$

as a relatively $\mathbb{Z}\oplus\mathbb{Z}$-graded group, where the integer m may be interpreted as a change of framing. More precisely, there exist explicit constants x and y and an identification

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q] \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

as graded $\mathbb{F}$-vector spaces so that

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right).$$

Proof. This amounts to a careful iterated application of the mapping cone for resolution of a positive crossing applied to the n positive crossings in $\tau(m+n)$. When $n=1$ we have

$$\widetilde{\mathrm{Kh}}(\tau(m+1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{k_\tau}{2},\tfrac{3k_\tau+2}{2}]\right)$$

where $k_\tau = c_\tau + m$. Set $[x,y] = [-\frac{k_\tau}{2},\frac{3k_\tau+2}{2}]$. Now when $n=2$ we obtain

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau(m+2)) &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{k_\tau+1}{2},\tfrac{3(k_\tau+1)+2}{2}]\right)\\ &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,1]\right)\end{aligned}$$

or, by unpacking the group $\widetilde{\mathrm{Kh}}(\tau(m+1))$ as in the previous case,

$$\begin{aligned}&\widetilde{\mathrm{Kh}}(\tau(m+2))\\ &\cong H_*\left(H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y]\right)[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,1]\right)\end{aligned}$$

as an iterated mapping cone. Said another way, this expression is simply the repeated application of the long exact sequence. This simplifies considerably however, since the two occurrences of the group $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$ appear in the same δ-grading. Since the differential of the mapping cone (or, the connecting homomorphism of the long exact sequence) raises δ-grading by one, there cannot be a differential between the copies of $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))$. As result,

$$\begin{aligned}
&\widetilde{\mathrm{Kh}}(\tau(m+2))\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-1,1] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}] \oplus \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,1]\right)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-1,1] \to \textstyle\bigoplus_{q=0}^{1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][-\tfrac{1}{2},\tfrac{1}{2}][0,q]\right)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{1}{2},\tfrac{1}{2}]
\end{aligned}$$

Now suppose for induction that

$$\widetilde{\mathrm{Kh}}(\tau(m+n-1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{n-2} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{n-2}{2},\tfrac{n-2}{2}]$$

and consider the group

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+n-1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{c}{2},\tfrac{3c+2}{2}]\right)$$

where $c = n_-(\tau(\frac{1}{0}))+n-1-n_-(\tau(m+n-1)) = n_-(\tau(\frac{1}{0}))-n_-(\tau(m))+n-1 = k_\tau+n-1$. Then

$$\begin{aligned}
&\widetilde{\mathrm{Kh}}(\tau(m+n))\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+n-1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{k_\tau+n-1}{2},\tfrac{3(k_\tau+n-1)+2}{2}]\right)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m+n-1))[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{k_\tau}{2},\tfrac{3k_\tau+2}{2}][0,n-1][-\tfrac{n-1}{2},\tfrac{n-1}{2}]\right)\\
&\cong H_*\left(H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{n-2} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{n-2}{2},\tfrac{n-2}{2}][-\tfrac{1}{2},\tfrac{1}{2}]\right.\\
&\qquad\qquad\qquad \left.\to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,n-1][-\tfrac{n-1}{2},\tfrac{n-1}{2}]\right)\\
&\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2},\tfrac{1}{2}] \to \textstyle\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]\right)[-\tfrac{n-1}{2},\tfrac{n-1}{2}]
\end{aligned}$$

noting once again that each of the occurrences of $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))$ differs only in the secondary grading.

Now as a relatively graded group, we are free to ignore the overall grading shift $[-\frac{n-1}{2}, \frac{n-1}{2}]$. Moreover, since $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, fixing an identification

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x+\tfrac{1}{2}, y-\tfrac{1}{2}][0,q] \cong \mathbb{F}[q]/q^n \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

we have that

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

as a relatively $\mathbb{Z} \oplus \mathbb{Z}$-graded group. □

Remark 5.2. *As stated, this lemma might be viewed from the point of Heegaard-Floer homology. In particular, the long exact sequence for integer surgeries may be stated*

$$\cdots \longrightarrow \widehat{\mathrm{HF}}(S^3_m(K)) \longrightarrow \widehat{\mathrm{HF}}(S^3_{m+n}(K)) \longrightarrow \bigoplus_n \widehat{\mathrm{HF}}(S^3) \longrightarrow \cdots$$

where

$$\bigoplus_n \widehat{\mathrm{HF}}(S^3) \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

when viewed with twisted coefficients (c.f. (Ozsváth and Szabó, 2008, Theorem 3.1)). We have given an analogous statement in terms of the Khovanov homology of the associated branch sets in the case when K is strongly invertible, a fact that is particularly interesting in light of Theorem 3.12.

Before turning to consequences of Lemma 5.1, we note that a similar statement is forced to exist for negative surgeries. Indeed, consider $\widetilde{\mathrm{Kh}}(\tau(m-n))$ for any integer m, and positive integer n. Setting $m' = m - n$ we have that

$$\widetilde{\mathrm{Kh}}(\tau(m')) \cong \widetilde{\mathrm{Kh}}(\tau(m-n))$$

and

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau(m)) &\cong \widetilde{\mathrm{Kh}}(\tau(m'+n))\\ &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m')) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)\\ &\cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m-n)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right).\end{aligned}$$

It follows that:

Lemma 5.3. *For any integer m, and positive integer n,*

$$\widetilde{\mathrm{Kh}}(\tau(m-n)) \cong H_*\left(\bigoplus_n \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0})) \to \widetilde{\mathrm{Kh}}(\tau(m))\right)$$

as a relatively $\mathbb{Z}\oplus\mathbb{Z}$-graded group, where the integer m may be interpreted as a change of framing. More precisely, there exist an explicit constants x' and y' (different than above) and an identification

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x',y'][0,q] \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

so that

$$\widetilde{\mathrm{Kh}}(\tau(m-n)) \cong H_*\left(\mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \to \widetilde{\mathrm{Kh}}(\tau(m))\right)$$

Remark 5.4. *In fact, it should be immediately clear that in this case the group*

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x',y'][0,q] \cong \mathbb{F}[q^{-1}]/q^{-n} \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

must lie in grading $\delta - 1$ relative to the group

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x+\tfrac{1}{2},y-\tfrac{1}{2}][0,q] \cong \mathbb{F}[q]/q^n \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

of Lemma 5.1 in grading δ. Alternatively, Lemma 5.3 may be proved by an argument nearly identical to the argument of Lemma 5.1, up to renaming constants.

5.2 Width stability.

There are two essential consequences that we derive from Lemma 5.1. Similar properties exist for branch sets associated to negative surgeries, although we will not state these, opting instead to pass to positive surgeries on the mirror to avoid negative coefficients.

Lemma 5.5. *For $N >> 0$ the exact sequence for $\widetilde{\mathrm{Kh}}(\tau(N+1))$ splits so that, ignoring gradings,*

$$\widetilde{\mathrm{Kh}}(\tau(N+1)) \cong \widetilde{\mathrm{Kh}}(\tau(N)) \oplus \mathbb{F}.$$

Proof. Let $N = m$ and $n = 1$ in the notation of Lemma 5.1, so that

$$\widetilde{\mathrm{Kh}}(\tau(N+1)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(N)) \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0})) \right)$$

On the other hand, with $m = 0$ and $n = N + 1$ we have that

$$\widetilde{\mathrm{Kh}}(\tau(N+1)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}[q]/q^{N+1} \right).$$

Since the differential preserves the secondary grading, for $N >> 0$ the generator represented by q^N cannot be in the image of the differential. □

Lemma 5.6. *Up to overall shift, the generators $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, when they survive in homology, are all supported in a single δ-grading.*

Proof. Immediate from the identification with the truncated polynomial ring in Lemma 5.1. □

As a result of Lemma 5.5, the width of the $\tau(n)$ may be calculated for all n once some finite collection of the values is known. Moreover, these quantities must be bounded, in light of Lemma 5.6.

Definition 5.7. *For a given strongly invertible knot and compatible associated quotient* *tangle, define*

$$w_{\max} = \max_{n \in \mathbb{Z}} \{w(\tau(n))\}$$

and

$$w_{\min} = \min_{n \in \mathbb{Z}} \{w(\tau(n))\}.$$

Lemma 5.8. *Suppose $w_{\min} = w(\tau(N))$ for $|N| >> 0$. Then either $w_{\min} = 1$ and $T = (B^3, \tau)$ is the tangle associated to the trivial knot, or $w_{\min} > 1$ in which case $w_{\min} = w_{\max}$.*

Proof. First suppose $w_{\min} = 1$, so that $w(\tau(N)) = 1$ for all $|N|$ sufficiently large. Then by Proposition 3.16, $S^3_{\pm N}(K) = \mathbf{\Sigma}(S^3, \tau(\pm N))$ must be an L-space for all N sufficiently large. However, by Proposition 3.36, K must be the trivial knot (and hence $\tau(0) \simeq \bigcirc \sqcup \bigcirc$).

Now suppose that $w_{\min} > 1$ for $|N| >> 0$, and choose m sufficiently negative so that $w(\tau(m)) = w_{\min}$. Then we have

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for all $n > 0$. In particular, since $w_{\min} = w(\tau(m+n)$ for n sufficiently large, it must be that $\mathrm{Supp}(\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]) \subset \mathrm{Supp}(\widetilde{\mathrm{Kh}}(\tau(m))$. As a result, a decrease in width would contradict our assumption that $w(\tau(m))$ is minimal, hence $w_{\min} = w_{\max}$. □

Lemma 5.9. *The maximum and minimum widths differ by at most 1. That is, either $w_{\max} = w_{\min}$ or $w_{\max} = w_{\min} + 1$.*

Proof. First notice that the statement holds for the tangle associate to the quotient of the trivial knot by Lemma 5.8, since $w(\tau(0)) = w(\bigcirc \sqcup \bigcirc) = 2$.

Assuming then that K is non trivial, without loss of generality we may suppose that $w_{\min} = w(\tau(N))$ for $N >> 0$ and that $w_{\max} = w(\tau(N))$ for $N << 0$. Now choosing m sufficiently negative in the notation of Lemma 5.1 we have that $w_{\max} = w(\tau(m))$.

Further,

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for every $n > 0$. Since $w_{\min} = w(\tau(m+n))$ for some n, the group $\mathbb{F}^n \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ must be in a fixed grading supported by $\widetilde{\mathrm{Kh}}(\tau(m))$. Therefore, if

$$\widetilde{\mathrm{Kh}}(\tau(m)) \cong \bigoplus_{\delta} \mathbb{F}^{b_\delta} \cong \mathbb{F}^{b_1} \oplus \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_{w_{\max}}}$$

then since the differential of the mapping cone raises δ-grading by one we have that $w_{\max} = w_{\min}$ unless

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\begin{pmatrix} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & \cdots & \mathbb{F}^{b_{w_{\max}}} \\ & \searrow & & & \\ & & \mathbb{F}^n & & \end{pmatrix}$$

wherein this case the possibility arises for $w_{\max} = w_{\min} + 1$. □

Remark 5.10. *We remark that, whenever $w_{\max} = w_{\min} + 1$ for a tangle associated to a non-trivial knot in S^3, there is a unique ℓ for which $w(\tau(\ell))$ and $w(\tau(\ell+1))$ differ. Moreover, we may assume up to taking mirrors that $\ell \geq 0$.*

We note that, having fixed $\ell \geq 0$ whenever $w_{\max} = w_{\min} + 1$, the width either expands or decays. More precisely, the width expands whenever

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_*\begin{pmatrix} \mathbb{F}^{b_1} & \mathbb{F}^{b_2} & \cdots & \mathbb{F}^{b_{w_{\min}}} & & \\ & & & & \searrow & \\ & & & & & \mathbb{F}^n \end{pmatrix}$$

and the possibility for width decay arises whenever

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_*\begin{pmatrix} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & \cdots & \mathbb{F}^{b_{w_{\max}}} \\ & \searrow & & & \\ & & \mathbb{F}^n & & \end{pmatrix}$$

for $m = 0$ in the notation of Lemma 5.1.

For example, Berge knots (chosen so that the lens space surgeries are positive) give rise to a family of tangles for which the width decays (c.f. Proposition 4.23).

5.3 On determinants and resolutions

In the arguments that follow, we will rely heavily on resolutions of terminal crossings (see Definition 4.14) in branch sets $\tau(\frac{p}{q})$ for which $S^3_{p/q}(K) = \mathbf{\Sigma}(B^3, \tau(\frac{p}{q}))$. As such, we remark that

$$\det(\tau(\tfrac{p}{q})) = |H_1(S^3_{p/q}(K);\mathbb{Z})| = p$$

for any $\frac{p}{q} \geq 0$ (in all cases, we deal with negative surgeries by passing to the mirror image). Moreover if $\tau(\frac{p_0}{q_0})$ and $\tau(\frac{p_1}{q_1})$ are the links resulting from resolution of the terminal crossing, then

$$\det(\tau(\tfrac{p}{q})) = p = p_0 + p_1 = \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))$$

by applying Property 4.13.

As a result, $\widetilde{\mathrm{Kh}}(\tau(\frac{p}{q}))$ may be studied by applying Proposition 2.14 to the resolutions $\tau(\frac{p_0}{q_0})$ and $\tau(\frac{p_1}{q_1})$ whenever $\frac{p}{q} > 1$. In the case $\frac{p}{q} \in (0,1)$ the same arguments work by using Proposition 2.15 when treating continued fractions of length $r = 2$: here $\det(\tau(\frac{p_0}{q_0})) = \det(\tau(0)) = 0$.

By Lemma 5.1 we have that

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for a specific identification of $\bigoplus_n \widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ as a graded group. As a result,

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(n)) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n-1)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{1}{0}))\right)$$

whenever $n > 1$, and

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(0))[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{1}{0}))\right)$$

where $\widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{1}{0})) = \widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))$ since the signature of the trivial knot is 0. In either case,

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n+1))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n+1))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n))\right)$$

as absolutely graded groups (where the fixed shifts are adjusted accordingly by $[-\frac{1}{2}]$ in the case of $\tau(0)$). Notice that these inclusions are equalities whenever $w(\tau(n)) = w(\tau(n+1))$, so that the inclusions are only relevant in the case when the width changes by one.

5.4 An upper bound for width

Proposition 5.11. *Let K be a strongly invertible knot in S^3, with canonical associated quotient tangle $T = (B^3, \tau)$. Then $w(\tau(\frac{p}{q}))$ is bounded above by $w_{\max}$ for all $\frac{p}{q} \in \mathbb{Q}$.*

Proof. The proof is similar to that of Proposition 4.23 establishing the upper bound of 2 for the width of a branch set associated to surgery on a Berge knot. Again, we suppose without loss of generality that $\frac{p}{q} \geq 0$, and proceed in 2 cases.

Case 1: $1 \leq \frac{p}{q}$

By its definition, $w_{\max}$ provides the upper bound for $w(\tau(n))$ for any n. This provides a base for induction in r, the length of the continued fraction representation $\frac{p}{q} = [a_1, a_2, \ldots, a_r]$.

First consider the case $\frac{p}{q} = [a_1, 2]$. Here we have

$$\det(\tfrac{p}{q}) = p = p_0 + p_1 = a_1 + a_1 + 1 = \det(\tfrac{p_0}{q_0}) + \det(\tfrac{p_1}{q_1})$$

where where $\frac{p_0}{q_0} = [a_1]$ corresponds to the 1-resolution of the terminal crossing, and $\frac{p_1}{q_1} = [a_1, 1] = [a_1 + 1]$ corresponds to the 0-resolution of the terminal crossing. In either case $w(\frac{p_0}{q_0}), w(\frac{p_1}{q_1}) \leq w_{\max}$, and by applying Proposition 2.14 we have

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right).$$

Moreover, according to Section 5.3 we have that either

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))\right)$$

as a consequence of Lemma 5.1. Therefore,

$$\begin{aligned} w(\tau[a_1, 2]) &= w(\tau(\tfrac{2a_1+1}{2})) \\ &\leq \max\{w(\tau\lfloor\tfrac{2a_1+1}{2}\rfloor), w(\tau\lceil\tfrac{2a_1+1}{2}\rceil)\} \\ &= \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \\ &\leq w_{\max}. \end{aligned}$$

The same statement holds for $\frac{p}{q} = [a_1, a_2]$. By iterating Proposition 2.14 we have

$$\begin{array}{ccc} \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_0}{q_0})) & \longrightarrow & \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p}{q})) \\ & & \downarrow \\ \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_0}{q_0})) & \longrightarrow & \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_1'}{q_1'})) \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ & & \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_1}{q_1})) \end{array}$$

where the connecting homomorphisms have been omitted. Once again, as a consequence of supports we conclude that

$$\begin{aligned} w(\tau[a_1,a_2]) &= w(\tau(\tfrac{a_1a_2+1}{2})) \\ &\leq \max\{w(\tau\lfloor\tfrac{a_1a_2+1}{2}\rfloor), w(\tau\lceil\tfrac{a_1a_2+1}{2}\rceil)\} \\ &= \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \\ &\leq w_{\max}. \end{aligned}$$

Now for induction in r, given $\frac{p}{q} = [a_1, a_2, \ldots, a_{r-1}]$ the inductive hypothesis is that $w(\tau(\frac{p}{q})) \leq w_{\max}$ and one of

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}])\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}+1]))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}+1]))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,a_2,\ldots,a_{r-1}])\right)$$

holds.

This being the case, we claim that

$$w(\tau[a_1,a_2,\ldots,a_{r-1},a_r]) \leq \max\left\{w(\tau[a_1,a_2,\ldots,a_{r-1}]), w(\tau[a_1,a_2,\ldots,a_{r-1}+1])\right\}.$$

By resolving the terminal crossing of $\tau(\frac{p}{q})$ and applying Proposition 2.14

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong \begin{cases} H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_0}{q_0})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_1}{q_1}))\right) & r \text{ even} \\ H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_1}{q_1})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_0}{q_0}))\right) & r \text{ odd} \end{cases}$$

so that in either case $w(\tau(\frac{p}{q})) \leq \max\{w(\tau(\frac{p_0}{q_0})), w(\tau(\frac{p_1}{q_1}))\}$ if $a_r = 2$. By induction in a_r we have that

$$w(\tau[a_1,a_2,\ldots,a_{r-1},a_r]) \leq \max\{w(\tau[a_1,a_2,\ldots,a_{r+1}]), w(\tau[a_1,a_2,\ldots,a_{r-1}+1])\}$$

by applying Property 4.13 together with the induction hypothesis on supports.

As a result, by induction in length we have that

$$w(\tau(\tfrac{p}{q})) \leq \left\{ w(\tau\lfloor\tfrac{p}{q}\rfloor), w(\lceil\tfrac{p}{q}\rceil) \right\} \leq w_{\max},$$

concluding the proof in this case.

Case 2: $0 < \frac{p}{q} < 1$

The proof in this case follows the same lines as the previous case, and differs only in passing from the case $r = 2$ to $r = 1$. Indeed, the argument here is identical, once we replace the use of Proposition 2.14 is with that of its degenerative counterpart, Proposition 2.15. This is due to the fact that, while the determinants remain additive under resolution, $\det(\tau\lfloor\frac{p}{q}\rfloor) = 0$ in this case.

To see that this is so, consider once again the case $\frac{p}{q} = [a_1, 2] = [0, 2]$. By applying Proposition 2.15 we have

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_* \left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))[-\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0})) \right).$$

Moreover, according to Section 5.3 we have that either

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))[-\tfrac{1}{2}]\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right)$$

or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0}))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1}))[-\tfrac{1}{2}]\right)$$

as a consequence of Lemma 5.1. Therefore,

$$\begin{aligned} w(\tau[a_1, 2]) &= w(\tau(\tfrac{2a_1+1}{2})) \\ &\leq \max\{w(\tau\lfloor \tfrac{2a_1+1}{2} \rfloor), w(\tau\lceil \tfrac{2a_0+1}{2} \rceil)\} \\ &= \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \\ &\leq w_{\max}. \end{aligned}$$

The same statement holds for $\frac{p}{q} = [a_1, a_2]$. By iterating Proposition 2.15 as in the previous case so that

$$\begin{aligned} w(\tau[a_1, a_2]) &= w(\tau(\tfrac{a_1 a_2+1}{2})) \\ &\leq \max\{w(\tau\lfloor \tfrac{a_1 a_2+1}{2} \rfloor), w(\tau\lceil \tfrac{a_1 a_2+1}{2} \rceil)\} \\ &= \max\{w(\tau(a_1)), w(\tau(a_1+1))\} \\ &\leq w_{\max}. \end{aligned}$$

□

Remark 5.12. *Case 2, when $\frac{p}{q} \in (0,1)$, will be present in all of the arguments that follow. However, in every setting this case simply amounts to replacing Proposition 2.14 with Proposition 2.15 in passing from half-integer (continued fractions of length 2) to integer surgeries, as in the above proof. Thus, we will restrict, without loss of generality, to the case $\frac{p}{q} \geq 1$ in the arguments below.*

5.5 A lower bound for width

Proposition 5.13. *Let K be a strongly invertible knot in S^3, with canonical associated quotient tangle $T = (B^3, \tau)$. If $w_{\max} = w_{\min}$ then $w(\tau(\frac{p}{q}))$ is bounded below by $w_{\min}$ for all $\frac{p}{q} \in \mathbb{Q}$.*

Proof. Without loss of generality, assume that $\frac{p}{q} \geq 1$.

Since $w_{\max} = w_{\min} = w$, we have that $w = w(\tau(n))$ for every $n \in \mathbb{Z}$. In particular,

$$\operatorname{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n+1))\right) = \operatorname{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(n))\right)$$

as a consequence of Lemma 5.1. Thus, applying Proposition 2.14

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[a_1,2]) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(a_1)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(a_1+1))\right)$$

so that if $\widetilde{\mathrm{Kh}}(\tau(a_1)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_w}$ and $\widetilde{\mathrm{Kh}}(\tau(a_1+1)) \cong \mathbb{F}^{b'_1} \oplus \cdots \oplus \mathbb{F}^{b'_w}$ (note that $b_i \neq b'_i$ for precisely one value $1 \leq i \leq w$) then

$$\widetilde{\mathrm{Kh}}(\tau[a_1,2]) \cong H_*\left(\begin{array}{ccccccc} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_w} \\ & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_w} \end{array}\right)$$

as a relatively graded group, since the differential of the mapping cone raises δ-grading by 1. Notice in particular that $b^*_1 \geq b'_1$ and $b^*_w \geq b_w$ for $\widetilde{\mathrm{Kh}}(\tau[a_1,2]) = \mathbb{F}^{b^*_1} \oplus \cdots \oplus \mathbb{F}^{b^*_w}$, so that $w(\tau[a_1,2]) = w$

Similarly, for $\frac{p}{q} = [a_1,a_2]$ in general, we may iteratively apply Proposition 2.14 $a_2 - 1$ times to the same end:

$$\widetilde{\mathrm{Kh}}([a_1,a_2]) \cong H_*\left(\begin{array}{c} \begin{array}{ccccccc} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_w} \\ \vdots & & \vdots & & & & \vdots \end{array} \\ H_*\left(\begin{array}{ccccccc} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_w} \\ & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_w} \end{array}\right) \end{array}\right)$$

so that $b^*_1 \geq b'_1$ and $b^*_w \geq b_w$ for $\widetilde{\mathrm{Kh}}(\tau[a_1,a_2]) = \mathbb{F}^{b^*_1} \oplus \cdots \oplus \mathbb{F}^{b^*_w}$, and once again $w(\tau[a_1,a_2]) = w$.

To complete the proof then, we induct in r with the assumption that $w(\tau(\frac{p}{q})) = w$ for all $\frac{p}{q} = [a_1, \ldots, a_{r-1}]$, and that

$$\text{Supp}\left(\widetilde{\text{Kh}}_\sigma(\tau[a_1, a_2, \ldots, a_{r-1}])\right) = \text{Supp}\left(\widetilde{\text{Kh}}_\sigma(\tau[a_1, a_2, \ldots, a_{r-1}+1]))\right)$$

holds.

This being the case, we claim that

$$w(\tau[a_1, a_2, \ldots, a_{r-1}, a_r]) \leq \max\left\{w(\tau[a_1, a_2, \ldots, a_{r-1}]), w(\tau[a_1, a_2, \ldots, a_{r-1}+1])\right\}.$$

Indeed, when $a_r = 2$ we have that

$$\widetilde{\text{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong \begin{cases} H_*\left(\widetilde{\text{Kh}}_\sigma(\tau(\frac{p_0}{q_0})) \to \widetilde{\text{Kh}}_\sigma(\tau(\frac{p_1}{q_1}))\right) & r \text{ even} \\ H_*\left(\widetilde{\text{Kh}}_\sigma(\tau(\frac{p_1}{q_1})) \to \widetilde{\text{Kh}}_\sigma(\tau(\frac{p_0}{q_0}))\right) & r \text{ odd} \end{cases}$$

by applying Proposition 2.14 so that in either case $w(\tau(\frac{p}{q})) = w(\tau(\frac{p_0}{q_0})), w(\tau(\frac{p_1}{q_1}))$ since the corresponding groups have the same support. By induction in a_r we have that

$$w(\tau[a_1, a_2, \ldots, a_{r-1}, a_r]) = w(\tau[a_1, a_2, \ldots, a_{r+1}]), w(\tau[a_1, a_2, \ldots, a_{r-1}+1])$$

as before, by applying the induction hypothesis on supports.

As a result, by induction in length we have that

$$w(\tau(\tfrac{p}{q})) = w,$$

concluding the proof. □

Combining Proposition 5.13 with Proposition 5.11 we have immediately that

$$w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$$

takes a single value $w \in \mathbb{N}$ when $w = w_{\max} = w_{\min}$, where $T = (B^3, \tau)$ is the canonical representative for the quotient tangle associated to a strongly invertible knot in S^3.

5.6 Expansion and decay

By Remark 5.10, if $w_{\max} = w_{\min}+1$ then there is a unique value ℓ, which we may assume is positive, for which either $w_{\min} = w(\tau(\ell)) < w(\tau(\ell+1)) = w_{\max}$ (width *expansion*) or $w_{\max} = w(\tau(\ell)) > w(\tau(\ell+1)) = w_{\min}$ (width *decay*).

In each setting, we establish a sufficient (though certainly not necessary) condition for which $w_{\min}$ still provides a lower bound for $w(\tau(\frac{p}{q}))$.

Definition 5.14. *T is expansion generic if $b_k > 1$ where*

$$\widetilde{\mathrm{Kh}}(\tau(\ell)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

so that $w_{\min} = k$ and

$$\widetilde{\mathrm{Kh}}(\tau(\ell+1)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k} \oplus \mathbb{F}$$

so that $w_{\max} = k+1$, where $k > 0$.

Definition 5.15. *T is decay generic if $b_1 > 1$ where*

$$\widetilde{\mathrm{Kh}}(\tau(\ell)) \cong \mathbb{F} \oplus \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

so that $w_{\max} = k+1$ and

$$\widetilde{\mathrm{Kh}}(\tau(\ell+1)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

so that $w_{\min} = k$, where $k > 0$.

Both of these notions are well defined, according Lemma 5.9.

Notice that if T is expansion generic, then $T^{\star}$ is decay generic, and vice versa. These both seem to be stronger conditions than necessary, however genericity (in each sense) turns out to be the rule rather than the exception when we turn to applications of homological width.

Proposition 5.16. *If T is expansion generic then $w(\tau(\frac{p}{q}))$ is bounded below by $w_{\min}$ for all $\frac{p}{q} \in \mathbb{Q}$.*

Proof. Let $w(\tau(\ell)) = k = w_{\min}$ and $w(\tau(\ell+1)) = k+1 = w_{\max}$. First notice that for $\frac{p}{q} \notin [\ell, \ell+1]$ the proof proceeds exactly as in the proof of Proposition 5.13. Thus we are left to consider the case when $\frac{p}{q} \in [\ell, \ell+1]$. Without loss of generality, we may assume that $\ell > 0$: if this is not the case, the argument below goes through with Proposition 2.15 replacing Proposition 2.14 where necessary, as is now familiar.

Now when $[a_1, a_2] = [\ell, 2]$, we have that

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,2]) \cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right)$$

by resolving the terminal crossing and applying Proposition 2.14. By applying Lemma 5.1, notice that

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right)$$

gives

$$\widetilde{\mathrm{Kh}}(\tau[\ell,2]) \cong H_*\left(\begin{array}{ccccccccc} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} & & \mathbb{F} \end{array}\right)$$

so that $w(\tau[\ell,2]) \geq k$ due to expansion genericity ($b_k > 1$), since this ensures that groups in gradings 1 and k survive in homology.

Now consider the case $\frac{p}{q} = [\ell, 3]$. Again, we have that

$$\begin{aligned} \widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,3]) &\cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,2])\right) \\ &\cong H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\ell+1))\right)\right) \end{aligned}$$

so that

$$\widetilde{\mathrm{Kh}}(\tau[\ell,3]) \cong H_* \left(\begin{array}{ccccccccc} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_k} & & \mathbb{F}^{\epsilon} \end{array} \right)$$

where $\epsilon = 0, 1$ arising from

$$\widetilde{\mathrm{Kh}}(\tau[\ell,2]) \cong \mathbb{F}^{b'_1} \oplus \mathbb{F}^{b'_2} \oplus \cdots \oplus \mathbb{F}^{b'_k} \oplus \mathbb{F}^{\epsilon}.$$

Note that $b'_k > 0$, since $w(\tau[\ell,2]) \geq k$. If $\epsilon = 0$ then groups survive in degrees 1 and k so the width is k; in the case $\epsilon = 1$, $w(\tau[\ell,3]) \geq k$ due to expansion genericity as before. Proceeding in this way by iterating Proposition 2.14, we obtain the desired result for all $\tau(\frac{p}{q})$ when $\frac{p}{q} = [\ell, a_2]$. Notice that either

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2])\right) = \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2+1])\right),$$

in which case the proof concludes along the lines of the proof of Proposition 5.13, or

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2])\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell,a_2+1])\right).$$

In the case of the latter, we remark that $\widetilde{\mathrm{Kh}}(\tau[\ell,a_2]) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$ and $\widetilde{\mathrm{Kh}}(\tau[\ell,a_2]) \cong \mathbb{F}^{b'_1} \oplus \cdots \oplus \mathbb{F}^{b'_k} \oplus \mathbb{F}^{b'_{k+1}}$with $b_k > b'_{k+1}$.

We now proceed by induction, assuming the result holds for continued fractions of length $r-1$, with the support the Khovanov homology of the zero resolution of the terminal crossing included in the support of the Khovanov homology of the one resolution (once the gradings have been shifted by the signatures, according to Proposition 2.14).

Now for $\frac{p}{q} = [\ell, a_2, \ldots, a_{r-1}, 2]$,

$$\widetilde{\mathrm{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong \begin{cases} H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_0}{q_0})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_1}{q_1}))\right) & r \text{ even} \\ H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_1}{q_1})) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\frac{p_0}{q_0}))\right) & r \text{ odd} \end{cases}$$

so that

$$\widetilde{\mathrm{Kh}}(\tau(\tfrac{p}{q})) \cong H_* \begin{pmatrix} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_k} & & \mathbb{F}^{b'_{k+1}} \end{pmatrix}$$

where $b_k > b'_{k+1}$. Therefore, since there must be non-trivial groups in the first and k^{th} gradings, $w(\tau(\frac{p}{q})) \geq k$. To conclude the proof then it remains only to iterate this argument in a_r, as in the case $r = 2$. □

Proposition 5.17. *If T is decay generic then $w(\tau(\frac{p}{q}))$ is bounded below by $w_{\min}$ for all $\frac{p}{q} \in \mathbb{Q}$.*

Proof. The proof is almost identical to the proof of Proposition 5.16.

Let $w(\tau(\ell)) = k + 1 = w_{\max}$ and $w(\tau(\ell + 1)) = k = w_{\min}$. Again, notice that for $\frac{p}{q} \notin [\ell, \ell+1]$ the proof proceeds exactly as in the proof of Proposition 5.13. Thus we are left to consider the case when $\frac{p}{q} \in [\ell, \ell + 1]$. Without loss of generality, we may assume that $\ell > 0$.

Now when $[a_1, a_2] = [\ell, 2]$ is a half-integer, we have that

$$\widetilde{\mathrm{Kh}}_\sigma(\tau[\ell, 2]) \cong H_* \left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell)) \to \widetilde{\mathrm{Kh}}_\sigma(\tau(\ell + 1))\right)$$

by resolving the terminal crossing and applying Proposition 2.14. Notice however that since

$$\mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell + 1))\right) \subseteq \mathrm{Supp}\left(\widetilde{\mathrm{Kh}}_\sigma(\tau(\ell))\right)$$

this gives

$$\widetilde{\mathrm{Kh}}(\tau[\ell, 2]) \cong H_* \begin{pmatrix} \mathbb{F} & & \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ & & \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_k} \end{pmatrix}$$

so that $w(\tau[\ell, 2]) \geq k$ due to expansion genericity ($b_1 > 1$), since this ensures that groups in gradings 1 and k survive in homology.

The conclusion then follows by induction in the length of the continued fraction associated to $\frac{p}{q}$, assuming the inclusion of supports as before. $\square$

Collecting the above results, we have that

$$w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$$

takes values $[w_{\min}, w_{\min}+1] \subset \mathbb{N}$ when in the decay or expansion generic setting, where $T = (B^3, \tau)$ is the canonical representative for the quotient tangle associated to a strongly invertible knot in S^3.

5.7 Lee's result, revisited

We now have all the material in place to see why 2-bridge knots have thin Khovanov homology, a result due originally to Lee, and key component of Theorem 4.24.

Since two-bridge knots arise as the branch sets of lens spaces, we need to consider surgery on the trivial knot in S^3; the associated quotient tangle is rational, and the canonical representative is $(B^3, \asymp)$ since $\det(\tau(0)) = \det(\bigcirc \sqcup \bigcirc) = 0$ (equivalently, $S^2 \times S^1 = \mathbf{\Sigma}(S^3, \bigcirc \sqcup \bigcirc)$).

Since both $\tau(\frac{1}{0})$ and $\tau(1)$ are the trivial knot, applying Lemma 5.1 we have that

$$\mathbb{F} \cong H_*(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}).$$

Recall that $\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F} \oplus \mathbb{F}$ as a relatively $\mathbb{Z}$-graded group. Now it follows that the branch sets corresponding to positive integer surgery have Khovanov homology

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong H_*(\widetilde{\mathrm{Kh}}(\tau(0)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]) \cong \mathbb{F}^n,$$

hence $w(\tau(n))$ is thin for all $n \neq 0$.[2]

---

[2] Of course, there is enough information here to work out $\widetilde{\mathrm{Kh}}(\tau(n))$ completely as an absolutely bi-graded group.

Without loss of generality, we consider $\widetilde{\mathrm{Kh}}(\tau(\frac{p}{q}))$ for $\frac{p}{q} > 0$. In fact, $\tau[0, a_2, a_3, \ldots, a_r] \simeq \tau[a_3, \ldots, a_r]$, so we need only consider $\frac{p}{q} \geq 1$. Now it is a quick application of Proposition 5.11 to see that $\tau(\frac{p}{q})$ is a thin link, for all $\frac{p}{q} \neq 0$, since $\tau(n)$ is thin for all $n \neq 0$.

Note that in constructing 2-bridge links in this way, we recover Schubert's normal form for this class (Schubert, 1956).

CHAPTER VI

SURGERY OBSTRUCTIONS FROM KHOVANOV HOMOLOGY.

We are now in a position to assemble the material developed to this point into obstructions to certain exceptional surgeries. In particular, we give obstructions to lens space surgeries and finite fillings from Khovanov homology, and we give a range of calculations as illustration of this application of Khovanov homology.

While these examples are essentially the content of (Watson, 2008b), the obstructions developed here represent a strengthening of the results found in that work. In particular, the results of this chapter do not depend on the cyclic surgery theorem (Theorem 1.25) or the related results of Boyer and Zhang (Theorem 1.26 and Theorem 1.27).

6.1 Width obstructions

Theorem 6.1. *Let $K \hookrightarrow S^3$ be strongly invertible with canonical associated quotient tangle $T = (B^3, \tau)$. Then $w(\tau(\frac{p}{q})) > 1$ implies that $S^3_{p/q}(K)$ is not a lens space, and $w(\tau(\frac{p}{q})) > 2$ implies that $S^3_{p/q}(K)$ has infinite fundamental group.*

Proof. For $w > 1$ the statement follows from Theorem 4.24; for $w > 2$ the statement follows from Theorem 4.25. □

Our aim is to show that this is an effective obstruction by applying the results of Chapter 5.

Let $T = (B^3, \tau)$ be the canonical representative for the tangle associated to a strongly invertible knot in S^3. Recall that $\tau(\frac{1}{0})$ is the trivial knot, and

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

for some explicit identification

$$\mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \cong \mathbb{F}[q]/q^n \cong \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q]$$

as a graded $\mathbb{F}$-vector space. Here, $n > 0$ and the fixed grading shift depends on the tangle, and the integer m (c.f. Lemma 5.1). If

$$\widetilde{\mathrm{Kh}}(\tau(m)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

as a relatively $\mathbb{Z}$-graded $\mathbb{F}$-vector space, so that $w(\tau(m)) = k$, then the graded vector space $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ is added to some fixed (relative) grading δ^+ for $1 \leq \delta^+ \leq k+1$.

In the situation that the width decays (c.f Definition 5.15), $\delta^+ = 2$, and in the situation that width expands, $\delta^+ = k+1$ (c.f. Definition 5.14). If the width neither decays nor expands then the tangle will be referred to as width stable.

Definition 6.2. *The tangle $T = (B^3, \tau)$ is generic if it is width stable, or if the width decays (respectively, expands) then it is decay generic as in Definition 5.15 (respectively expansion generic as in Definition 5.14).*

A much stronger form of genericity exists, and will be useful in application.

Proposition 6.3. *If for each δ-grading supporting $\widetilde{\mathrm{Kh}}(\tau(m))$, for any m, there is a q-grading for which $\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}(\tau(m)) > \operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}_q(\tau(m)) > 1$, then the associated quotient tangle is generic.*

Proof. This is immediate from Lemma 5.1: since the graded vector space $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ has a unique generator in each secondary grading q, the condition $\operatorname{rk}\widetilde{\mathrm{Kh}}^{\delta}_q(\tau(m)) > 1$ ensures that $b_\delta \neq 0$ in $\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong \bigoplus_{\delta=1}^{k} \mathbb{F}^{b_\delta}$, for all n. As a result, the tangle is either width stable, or it is expansion generic as a result of $\operatorname{rk} \widetilde{\mathrm{Kh}}^{\delta}(\tau(m)) > \operatorname{rk} \widetilde{\mathrm{Kh}}^{\delta}_{q}(\tau(m))$. □

Our main results then are the following:

Theorem 6.4. *Let $K \hookrightarrow S^3$ be strongly invertible with generic associated quotient tangle. Then $w_{\min} > 1$ implies that K does not admit non-trivial lens space surgeries. Moreover, determining $w_{\min}$ is a finite check.*

Theorem 6.5. *Let $K \hookrightarrow S^3$ be strongly invertible with generic associated quotient tangle. Then $w_{\min} > 2$ implies that K does not admit non-trivial finite fillings. Moreover, determining $w_{\min}$ is a finite check.*

Remark 6.6. *In practice, one group $\widetilde{\mathrm{Kh}}(\tau(m))$ is enough to determine $w_{\min}$ and apply these obstructions.*

In the absence of the genericity hypothesis, the width is still a useful obstruction: In light of Theorem 1.25 it is enough to check the integer fillings of K when the question of lens space surgeries is of interest. Similarly, in the case of finite fillings only the integer and half-integer surgeries need to be considered in light of Theorem 1.26. In practice, however, genericity is easy to check and seems to be a relatively standard property. Indeed, the only example of a tangle failing this condition that this author has encountered in examples is given by rational tangles, that is, the tangle associated to the trivial knot. In the generic setting (see examples given below), it is particularly interesting that Khovanov homology is able to give useful surgery obstructions, without relying on these powerful theorems.

6.2 On constructing quotients

With the above in place, calculating width obstructions is straightforward, consisting of essentially three steps: realize a strong inversion on a knot, construct the quotient, and compute the Khovanov homology of the branch set for some integer surgery on the knot. The final step is direct calculation and, assuming the first step is done the second presents the only challenge. It is not difficult to construct this quotient, though requires some patience and attention.

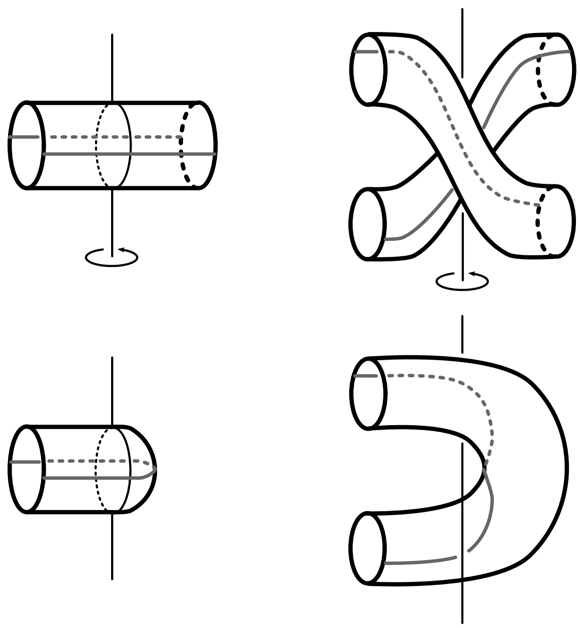


Figure 6.1 The local behaviour for quotients of strongly invertible knot complements. Notice that the quotient of a crossing across the axis of symmetry gives rise to a clasp between the image of the fixed point set and the quotient of the boundary.

To determine a fundamental domain for the action of the fixed involution, it suffices to 'cut' the knot complement along the axis of symmetry, and then apply the rules given in Figure 6.1. This is best displayed in examples (see below), but is expanded on in detail in (Bleiler, 1985; Montesinos and Whitten, 1986; Zimmermann, 1997), for example.

6.3 A first example: surgery on the figure eight

It is well known that the figure eight knot $K = 4_1$ does not admit lens space surgeries. In fact, Thurston classified the non-hyperbolic fillings of $S^3 \smallsetminus \nu(K)$ and showed that, aside from the trivial surgery, they all have infinite fundamental group (Thurston, 1980). That K does not admit (non-trivial) lens space surgeries has been reproved using the machinery of $SU(2)$ representation spaces (Kirk and Klassen, 1990; Klassen, 1991), essential laminations (Delman, 1995), character varieties (Tanguay, 1996) and most recently, Heegaard-Floer homology (Ozsváth and Szabó, 2005b). As a first example of the width obstructions developed here, we endeavour to add Khovanov homology to this list.

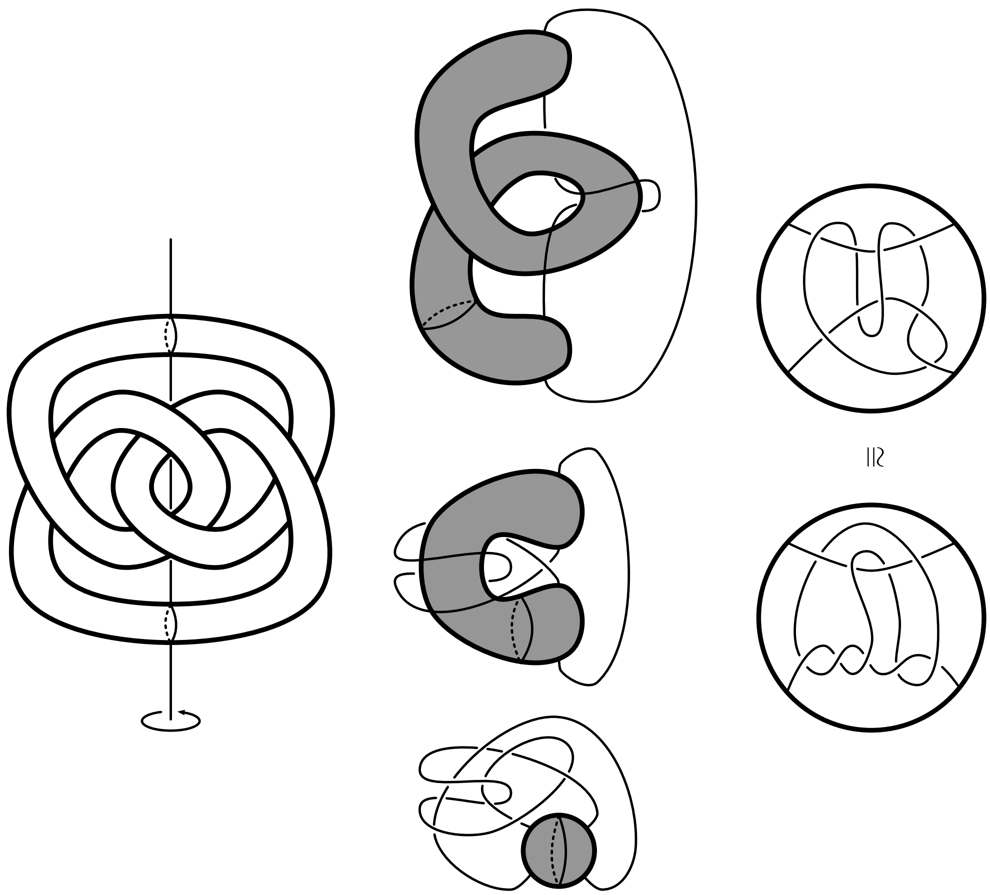


Figure 6.2 The strong inversion on the figure eight (left); isotopy of a fundamental domain (centre); and two representatives of the associated quotient tangle (right).

K is a strongly invertible knot, and this symmetry is shown in Figure 6.2 together with the associated quotient tangle. We have given two equivalent views of the associated quotient tangle. The first of these shows that the branch sets for integer surgeries may be expressed as closed 3-braids. For

$$\beta_n = \sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-4+n}$$

we have that $\tau(n) \simeq \overline{\beta_n}$, the closure of β_n. The Khovanov homology $\widetilde{\mathrm{Kh}}(\tau(-1))$, $\widetilde{\mathrm{Kh}}(\tau(0))$ and $\widetilde{\mathrm{Kh}}(\tau(+1))$ is given in Figure 6.3 (in particular, $\chi\left(\widetilde{\mathrm{Kh}}(\tau(0))\right) = \det(\tau(0)) = 0$). Notice that $w_{\min} = 2$ and that the tangle is decay generic. It follows at once that K does not admit lens space surgeries, and it seems worth pointing out that this result could have been inferred simply by inspection of the single Khovanov homology group $\widetilde{\mathrm{Kh}}(\tau(0))$.

		1
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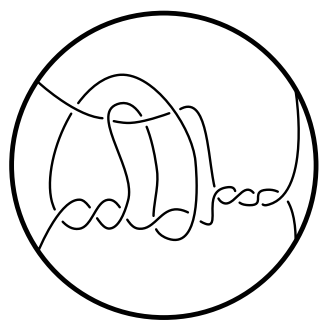


Figure 6.3 The canonical representative for the associated quotient tangle $T = (B^3, \tau)$ of the figure eight, and the reduced Khovanov homology groups $\widetilde{\mathrm{Kh}}(\tau(-1))$, $\widetilde{\mathrm{Kh}}(\tau(0))$ and $\widetilde{\mathrm{Kh}}(\tau(1))$ (from left to right). The δ^+ grading has been highlighted, in accordance with Lemma 5.1 setting $m = 0$.

More generally, we may use Lemma 5.1 to calculate:

Proposition 6.7.

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong \begin{cases} \mathbb{F}^{4+n} \oplus \mathbb{F}^4 & n > 0 \\ \mathbb{F} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4 & n = 0 \\ \mathbb{F}^{|n|} \oplus \mathbb{F}^4 \oplus \mathbb{F}^4 & n < 0 \end{cases}$$

Proof. The grading δ^+ is identified in Figure 6.3. By calculating that $\widetilde{\mathrm{Kh}}(\tau(-2)) \cong$

$\mathbb{F}^2 \oplus \mathbb{F}^4 \oplus \mathbb{F}^4$, Lemma 5.1, together with the groups

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau(-1)) &\cong \mathbb{F} \oplus \mathbb{F}^4 \oplus \mathbb{F}^4\\ \widetilde{\mathrm{Kh}}(\tau(0)) &\cong \mathbb{F} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4\\ \widetilde{\mathrm{Kh}}(\tau(1)) &\cong \qquad \mathbb{F}^5 \oplus \mathbb{F}^4\end{aligned}$$

forces the result. □

In fact, we have enough to recover Thurston's result:

Theorem 6.8. *Khovanov homology detects that the figure 8 admits no finite fillings.*

Proof. First notice that $w(\tau(n)) = 3$ for $n \leq 0$. As a result, a finite filling cannot arise by negative surgery on the figure eight. However, since the figure eight knot is amphicheiral, the same must be true for positive surgeries. □

Remark 6.9. *This result also follows quickly from Heegaard-Floer homology by applying Theorem 3.22, since the figure eight is alternating and hyperbolic, together with the fact that manifolds with finite fundamental group are L-spaces. Note that the Alexander polynomial for this knot is* $-t^{-1} + 3 - t$ *(c.f. Theorem 3.21).*

6.4 Some pretzel knots that do not admit finite fillings

According to Mattman (Mattman, 2000), it is unknown if the $(-2, p, q)$-pretzel knots admit fillings with finite fundamental group for $q \geq p \geq 5$. When $p = q = 5$ we have the following.

Theorem 6.10. *The* $(-2, 5, 5)$*-pretzel knot does not admit finite fillings.*

Proof. We begin by noting that the $(-2, 5, 5)$-pretzel knot, K, is strongly invertible in two ways as indicated in Figure 6.4. We will make use of the inversion indicated by the

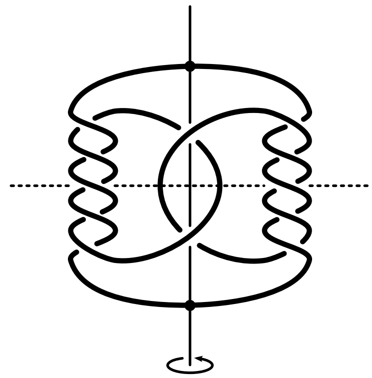


Figure 6.4 Two strong inversions on the $(-2, 5, 5)$-pretzel knot.

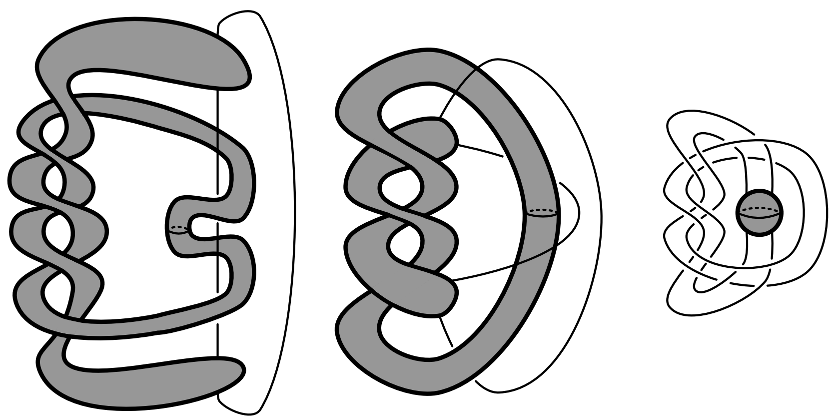


Figure 6.5 Isotopy of the fundamental domain for a strong inversion on the $(-2, 5, 5)$-pretzel knot. Notice that the resulting tangle has the property that integer closures are representable by closed 4-braids.

solid vertical line; the associated quotient tangle is calculated in Figure 6.5. Notice that the associated quotient tangle in this case gives rise to an obvious collection of 4-braids giving the branch sets for integer fillings. Setting

$$\beta_n = \sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2\sigma_1^{14+n}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})^3$$

we have $\tau(n) = \overline{\beta_n}$ by verifying that $\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F}^{16} \oplus \mathbb{F}^{20} \oplus \mathbb{F}^4$ so that $\det(\tau(0)) = 0$. The homologies of $\tau(n)$ for $n = -18, -17, -16, -15, -14$ are given in Figure 6.6. This

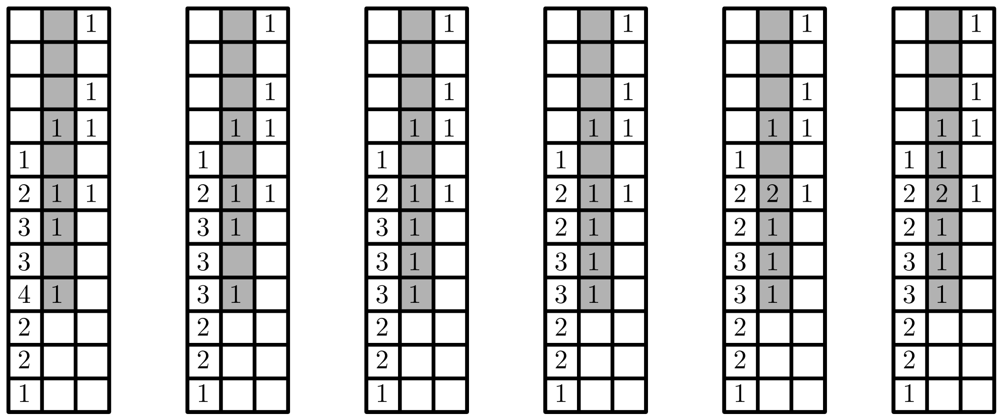


Figure 6.6 $\widetilde{\mathrm{Kh}}(\tau(n))$ for $n = -18, -17, -16, -15, -14$ (from left to right).

data is enough to infer that

$$\widetilde{\mathrm{Kh}}(\tau(n)) \cong \begin{cases} \mathbb{F}^{-n} \oplus \mathbb{F}^4 \oplus \mathbb{F}^4 & n < -16 \\ \mathbb{F}^{17} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4 & n < -16 \\ \mathbb{F}^{16} \oplus \mathbb{F}^{20+n} \oplus \mathbb{F}^4 & n > -16 \end{cases}$$

as relatively $\mathbb{Z}$-graded groups. In particular, $w_{\min} = w_{\max} = 3$ and the associated quotient tangle is generic.

The result now follows from Theorem 6.5. $\square$

Considering the same involution on the $(-2, p, p)$-pretzel knot K_p for all $p \geq 5$ we have that, in terms of the canonical associated quotient tangle, $\tau_p(n) = \overline{\beta_{n,p}}$ where

$$\begin{aligned} \beta_{n,p} &= \sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2\sigma_1^{14+4(\frac{p-5}{2})+n}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})^{p-2} \\ &= \sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2\sigma_1^{4+2p+n}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})^{p-2} \end{aligned}$$

so that $S^3_n(K_p) \cong \mathbf{\Sigma}(S^3, \tau_p(n))$. Notice that this expression changes only the number of double-strand full-twists in the associated quotient tangle (see Figure 6.5). From this expression, we were able to calculate $\widetilde{\mathrm{Kh}}(\tau_p(-4-2p))$ for p odd in the range $5 \leq p \leq 31$.[1] These calculations yield generic tangles in each case, with $w(\tau_p(-4-2p)) = p-2$, from which we conclude:

Theorem 6.11. *The $(-2,p,p)$-pretzel knots do not admit finite fillings for $5 \leq p \leq 31$.*

In fact, these calculations indicate a strong pattern from which one might guess that $\widetilde{\mathrm{Kh}}(\tau_p(0))$ has graded groups of rank

$$(N+12, 2N+12, N, N, N-8, N-8, \ldots, 12, 12, 4, 4)$$

where $N = 4 + 8(p-5)$, for every odd $p \geq 5$ (note that for $p = 5$ we have ranks $(16, 20, 4)$). In particular, it seems reasonable to conjecture that $w = p-2$ for the branch sets associated to surgery on K_p, so that Khovanov homology obstructs finite fillings on this class of knots.

We do not pursue this here, since the result may be shown by other means. Indeed, it is possible to use Theorem 3.21 to rule out L-space surgeries by considering the Alexander polynomials of the $(-2,p,p)$-pretzel knots,[2] and this has been carried out very recently by Ichihara and Jong completing Mattman's classification of Montesinos knots admitting finite fillings (Ichihara and Jong, 2008). Since then the result has received a different treatment by Futer, Ishikawa, Kabaya, Mattman and Shimokawa (Futer et al., 2008).

We remark that Mattman's classification (Mattman, 2000) using character variety methods illustrates some subtleties. Indeed, the $(-2,3,q)$-pretzel knots admit L-space surgeries for all $q \geq 3$ (see Theorem 3.28). Despite this fact however, Mattman shows that for $q > 9$ none of these manifolds can have finite fundamental group. On the other hand, for the $(-2,p,p)$-pretzel knots the character variety methods of Mattman were inconclusive, but this is precisely the setting in which Heegaard-Floer homology – and,

[1] When $p = 31$ this illustrates the limits of available computational tools: the resulting branch set has 140 crossings and reduced Khovanov homology of rank 1850.

[2] This was pointed out to the author by M. Hedden.

as seen here, Khovanov homology – obstructs finite fillings.

6.5 Khovanov homology obstructions in context: a final example

In light of the discussion above, it is natural to put the obstructions from Khovanov homology in contrast with those coming from Heegaard-Floer homology. The latter theory gives very stringent restrictions for the knot Floer homology of a knot admitting an L-space surgery (Ozsváth and Szabó, 2005b), and in particular the quickly implemented obstruction from the Alexander polynomial of Theorem 3.21. Since manifolds with finite fundamental group are known to be L-spaces (Proposition 3.18), this gives a useful obstruction to finite fillings. However, the criteria given in Theorem 3.21 can fail.[3] For example,

$$\Delta_K(t) = t^{-3} - t^{-2} + t^{-1} - 1 + t - t^2 + t^3$$

where K is the 14 crossing, non-alternating knot shown in Figure 6.7. Since this is a

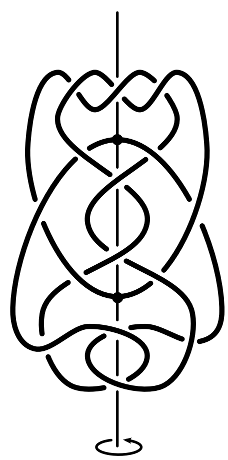


Figure 6.7 The strongly invertible knot $K = 14^n_{11893}$ has Alexander polynomial $\Delta_K(t) = t^{-3} - t^{-2} + t^{-1} - 1 + t - t^2 + t^3$.

[3]Though it rarely does: of the 27436 non-alternating 14-crossing knots, this obstruction fails on the order of 60 times. Among these knots, fewer still are strongly invertible.

strongly invertible knot, we are in a position to apply width obstructions from Khovanov homology, and to point out a particularly useful computational technique. The associ-

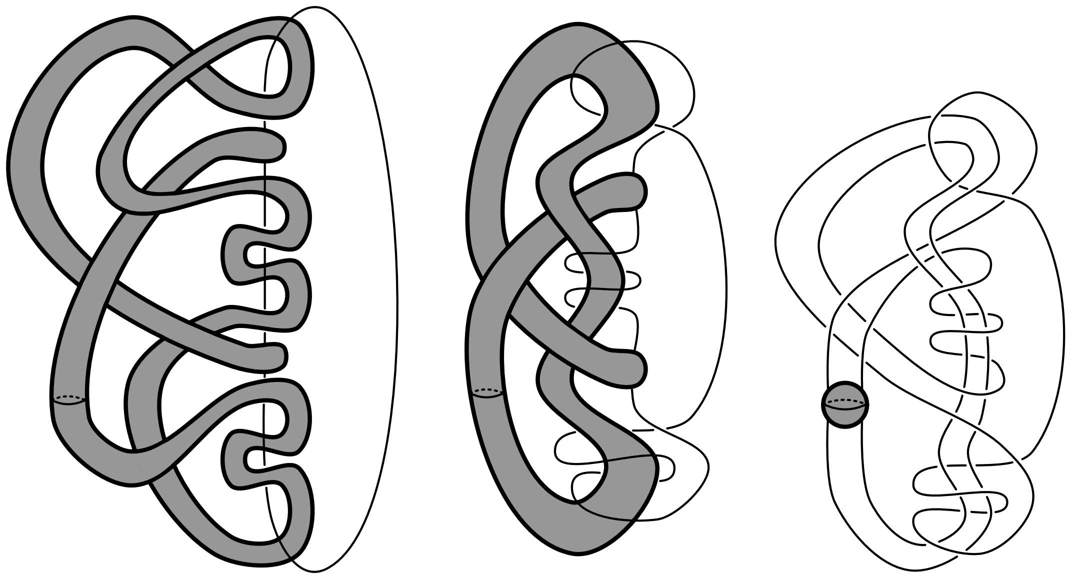


Figure 6.8 Isotopy of a fundamental domain for the involution on the complement of 14^n_{11893}.

ated quotient tangle is determined in Figure 6.8: notice that by construction the trivial knot $\tau(\frac{1}{0})$ is obtained by connecting the endpoints of the arcs of τ with two horizontal arcs inside the small sphere shown. Therefore, without knowing the framing, we can be sure that the branch sets for integer surgeries result from adding vertical half-twists inside the sphere, as shown in Figure 6.9.

Note that by doing this we have avoided incurring possible errors in further simplifying the tangle, and inspection of the resulting group immediately gives that the associated quotient tangle is generic, and the width is at least 4, for all n. As a result, we conclude:

Theorem 6.12. 14^n_{11893} *does not admit finite fillings; one Khovanov homology group suffices.*

In this setting, by switching the circled crossing of Figure 6.9 from positive to negative,

			1
			2
			2
		2	3
		4	3
		5	2
	2	7	2
	4	7	1
	5	6	
1	6	5	
2	7	3	
3	5		
3	4		
4	3		
3			
2			
2			

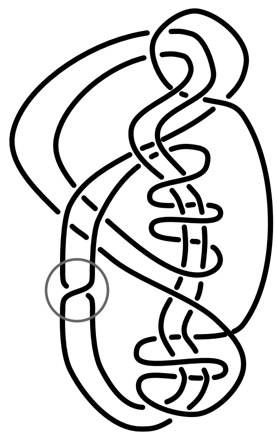


Figure 6.9 The branch set for some some integer surgery $S^3_n(K)$. Note that $\widetilde{\mathrm{Kh}}(\tau(n)) \cong \mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{39} \oplus \mathbb{F}^{16}$ so that $\chi = 59 - 52 = 7$ and $n = \pm 7$.

we can determine that

$$\widetilde{\mathrm{Kh}}(\tau(-9)) \cong \mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{41} \oplus \mathbb{F}^{16}$$
$$\widetilde{\mathrm{Kh}}(\tau(-7)) \cong \mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{39} \oplus \mathbb{F}^{16}$$

so that $w_{\min} = w_{\max} = 4$, and T is generic in the strong sense of Proposition 6.3, determining the width for the branch set of *any* surgery on K.

While it is possible that the full knot Floer homology of K obstructs L-space surgeries, this example shows that in certain settings the Khovanov homology obstructions may be more convenient from a computational standpoint when the question of finite fillings is of interest. Further, these obstructions may allow one to rule out finite fillings among L-spaces, a distinction that can be subtle.

CHAPTER VII

KHOVANOV HOMOLOGY AND THE TWO-FOLD BRANCHED COVER, REVISITED.

Mutation provides an easy method for producing distinct knots sharing a common two-fold branched cover: The mutation in the branch set corresponds to a trivial surgery in the cover. Due to a result of Wehrli (Wehrli, 2007), this provides a range of examples of manifolds that branch cover S^3 in more than one way, but for which the distinct branch sets have identical rank[1] in their respective Khovanov homology groups over $\mathbb{F}$.

From this point of view this fact is not completely surprising, given that Khovanov homology is closely related to the Heegaard-Floer homology of two-fold branched covers. More generally however, the following question has been posed by Ozsváth: is Khovanov homology an invariant of the two-fold branched cover? More precisely, Ozsváth's question asks if the total rank of the reduced Khovanov homology is an invariant of the two-fold branched cover. This chapter gives a negative answer by constructing manifolds that are two-fold branched covers of S^3 in two different ways where the two branch sets are distinguished by the total rank of their Khovanov homology.

The examples given here are all Seifert fibered, and were given in (Watson, 2008a). Hyperbolic examples seem difficult to obtain, and we give some constructions of infinitely many manifolds that branch in two different ways, with branch set that is distinguished

[1] In fact, the full Khovanov homology group of each mutant is the same, according to (Wehrli, 2007), although the question remains open in the case of $\mathbb{Z}$-coefficients. Infinite families of mutants with identical Khovanov homology (without restricition on coefficients) are produced in (Watson, 2007).

by Khovanov homology, but for which the total rank is the same. As a result, such examples are non-mutant, and serve as an illustration of Lemma 5.1 as a calculation tool.

7.1 Seifert fibered two-fold branched covers

Throughout this section, let K be the positive $(2,5)$-torus knot. In general, $T_{p,q}$ will denote the positive (p,q) torus knot in S^3, so that $K = T_{2,5}$.

Proposition 7.1. *$S^3_{\pm 1/n}(K)$ is Seifert fibered with base orbifold $S^2(2,5,10n \mp 1)$.*

Proof. Let $M = S^3 \smallsetminus \nu(K)$ so that $M(\alpha) = S^3_{p/q}(K)$ for $\alpha = p\mu + q\lambda$. Let ϕ denote a regular fibre in ∂M; it is well known that $\phi = 10\mu + \lambda$ (see (Moser, 1971), for example).

Now M is Seifert fibered with base orbifold $D^2(2,5)$, and $M(\alpha)$ is Seifert fibered with base orbifold $S^2(2,5,\Delta(\alpha,\varphi))$ whenever $\alpha \neq \varphi$, according to Theorem 1.21.

In the present setting, $\alpha = \pm\mu + n\lambda$ for $n > 0$ so that $M(\alpha) = S^3_{\pm 1/n}(K)$. Therefore,

$$\Delta(\alpha,\varphi) = |(\pm\mu + n\lambda)\cdot(10\mu+\lambda)| = \begin{cases} 10n-1 & \text{for positive surgeries} \\ 10n+1 & \text{for negative surgeries} \end{cases}$$

As a result, $M(\pm\mu+n\lambda) = S^3_{\pm 1/n}(K)$ is Seifert fibered with base orbifold $S^2(2,5,10n\mp 1)$ as claimed. □

Proposition 7.2. *$S^3_{\pm 1/n}(K) \cong \mathbf{\Sigma}(S^3, T_{5,10\mp 1})$.*

Proof. The Seifert structure on $S^3_{\pm 1/n}(K)$ is unique (see Proposition 1.19 or Proposition 1.23) and as a result this manifold must be homeomorphic to the Brieskorn sphere $\mathbf{\Sigma}(T_{5,10\mp 1})$ of Proposition 1.18. □

Since K is strongly invertible, there must be a second involution on $S^3_{\pm 1/n}(K)$ arising by extending the involution to the Dehn surgery. This corresponds to a Montesinos knot, constructed as follows.

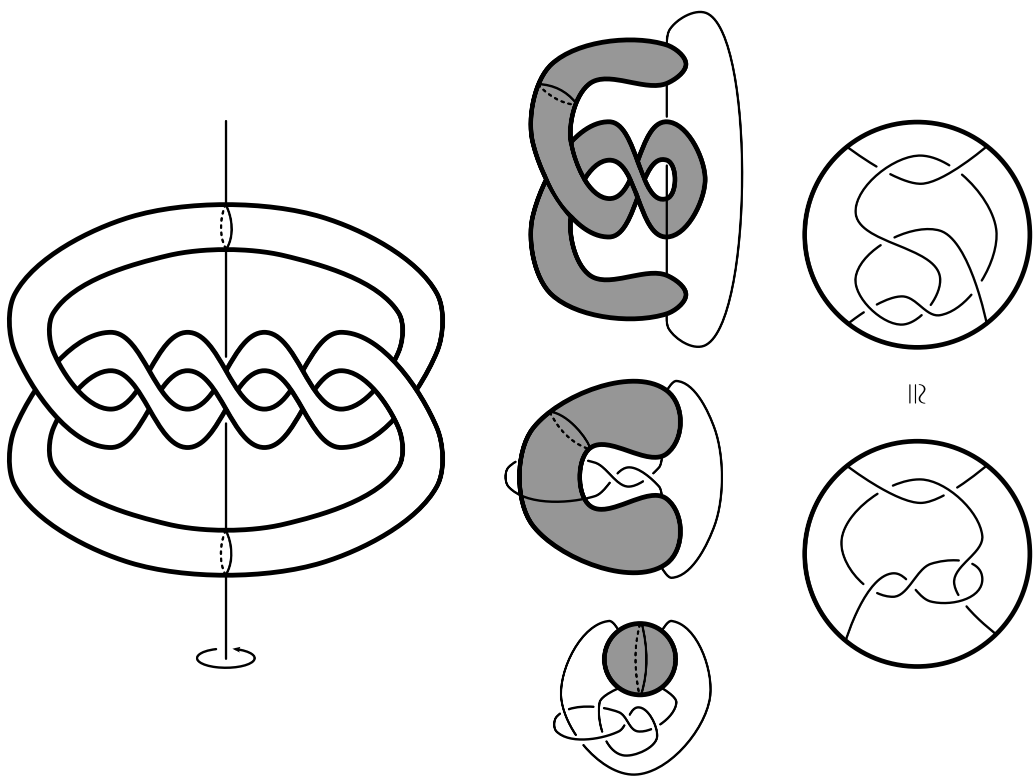


Figure 7.1 The strong inversion on the cinqfoil K (left); isotopy of a fundamental domain (centre); and two representatives of the associated quotient tangle (right). Notice that the Seifert fibre structure on the complement of K is reflected in the sum of rational tangles of the associated quotient tangle.

First, the associated quotient tangle is determined by isotopy of a fundamental domain for the fundamental for the action. This is shown in Figure 7.1. We must fix the canonical representative for the associated quotient tangle, and this is shown in Figure 7.2. Note that the knot $\tau(\frac{1}{0})$ is trivial, so we need only ensure that $\tau(0)$ gives a branch set for the zero surgery on K There are two ways to see that this is the case. First recall that $S^3_{10}(K)$ is a connect sum of lens spaces (see (Moser, 1971), for example). This is reflected in the numerator closure of either representative shown in Figure 7.1 as a connect sum of two-bridge knots. Alternatively, it suffices to check that $\chi\left(\widetilde{\mathrm{Kh}}(\tau(0))\right) = \det(\tau(0)) = 0$ (see Figure 7.3 below).

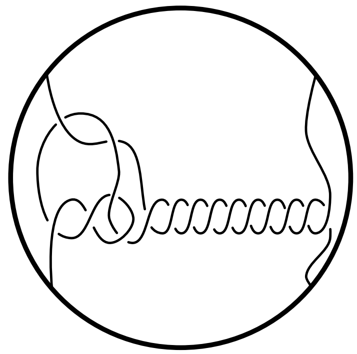


Figure 7.2 The canonical representative of the associated quotient tangle for the cinqfoil, K.

Proposition 7.3. $\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(\frac{1}{n})) \leq 16n-1$ *and* $\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(-\frac{1}{n})) \leq 16n+1$.

Proof. First note that $\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(\pm 1)) = 16 \mp 1$, as shown in Figure 7.3. The result follows by induction in n: by applying the long exact sequence for Khovanov homology we have that

$$\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{n}) \leq \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{n-1})) + \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(0)) = \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{n-1})) + 16$$

and

$$\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(-\tfrac{1}{n}) \leq \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(-\tfrac{1}{n-1})) + \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(0)) = \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(-\tfrac{1}{n-1})) + 16.$$

□

By construction, we have that

$$S^3_{\pm 1/n}(K) \cong \mathbf{\Sigma}(S^3, \tau(\pm\tfrac{1}{n})) \cong \mathbf{\Sigma}(S^3, T_{5,10n\mp 1}).$$

Further, direct calculation shows that $\operatorname{rk}\widetilde{\mathrm{Kh}}(T_{5,10\pm 1}) = 65 \pm 8$ and $\operatorname{rk}\widetilde{\mathrm{Kh}}(T_{5,20\pm 1}) = 257 \pm 16$. As a result, we have the following:

Example 7.4. *The Seifert fibered spaces* $S^3_{-1/2}(K)$, $S^3_{-1}(K)$, $S^3_{1}(K)$ *and* $S^3_{1/2}(K)$ *each*

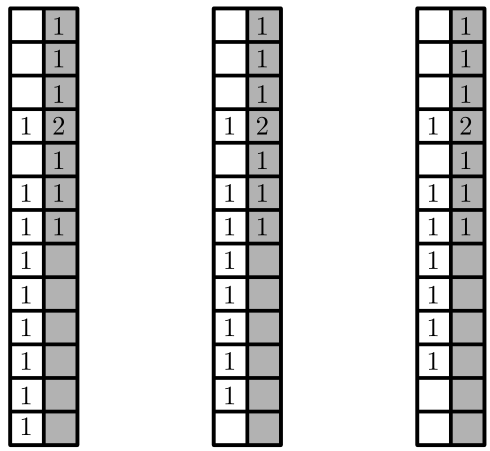


Figure 7.3 The reduced Khovanov homology of $\tau(-1)$ (left), $\tau(0)$ (centre), and $\tau(1)$ (right). Notice that $\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F}^8 \oplus \mathbb{F}^8$ implies that $\det(\tau(0)) = 0$.

branch cover S^3 in two ways. Moreover, the rank of the reduced Khovanov homology distinguishes the pair of branch sets in each of the four cases.

Corollary 7.5. *The total rank of the reduced Khovanov homology is not an invariant of the two-fold branched cover.*

These examples show that the *Seifert* and *Montesinos* involutions on Seifert fibered homology spheres may be distinguished by the rank of Khovanov homology. Experimental evidence suggests that the rank of the Khovanov homology for torus knots grows at a rate that is *at least* linear. As such it seems safe to make the following conjecture:

Conjecture 7.6. *The Seifert and Montesinos involutions are distinguished by the rank of Khovanov homology for Seifert fibered homology spheres obtained by surgery on the cinqfoil.*

While this is certainly not the case for surgery on the trefoil,[2] it seems likely that further examples may be obtained by considering surgery on $T_{2,2n+1}$ for $n > 2$.

[2]Indeed, the Seifert and Montesinos involution coincide for $+1$-surgery on the trefoil (see Theorem 1.14).

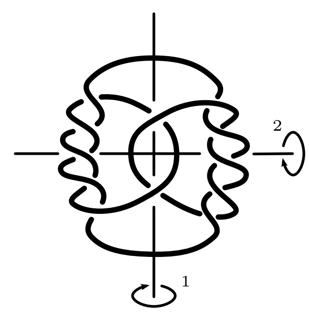
7.2 Hyperbolic two-fold branched covers.

A natural question is whether a similar example exists in the hyperbolic setting.

Question 7.7. *Does there exist a hyperbolic two-fold branched cover of S^3, branching in more than one way, so that the rank of Khovanov homology distinguishes the branch sets?*

We remark that a hyperbolic two-fold branched cover can have at most 9 non-equivalent branch sets (Reni, 2000, Corollary 1). Indeed, it has been shown that this bound is realized (Kawauchi, 2006).

7.2.1 Pretzel knots, revisited.



As we have seen, the $(-2,5,5)$-pretzel knot is strongly invertible in two distinct ways. By considering surgery on this knot then we obtain two (possibly distinct) branch sets for the resulting manifold as a two-fold branched cover of S^3. We have determined the associated quotient tangle for one of the two involutions shown in Section 6.4, and the second tangle may be determined by the same method. Both tangles are shown in Figure 7.4, though not with canonical framing.

Canonical framings are obtained by adding 14 and 22 (positive) half twists to the diagrams of T_1 and T_2 shown, respectively. As a result, we compute $\widetilde{\mathrm{Kh}}(\tau_1(0)) \cong \mathbb{F}^{16} \oplus \mathbb{F}^{20} \oplus \mathbb{F}^{4}$, and have $\widetilde{\mathrm{Kh}}(\tau_2(0)) \cong \mathbb{F}^{4} \oplus \mathbb{F}^{20} \oplus \mathbb{F}^{16}$ from Section 6.4. These groups are shown in Figure 7.5

More generally, from the behaviour of these groups we have that

$$\widetilde{\mathrm{Kh}}(\tau_1(n)) \cong \mathbb{F}^{16} \oplus \mathbb{F}^{20+n} \oplus \mathbb{F}^{4}$$

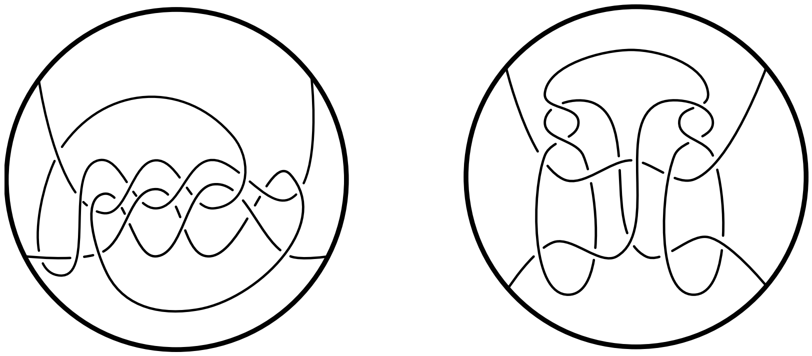


Figure 7.4 The tangles T_1 (left) and T_2 (right) associated to the distinct involutions on the $(-2, 5, 5)$-pretzel

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Figure 7.5 The groups $\widetilde{\mathrm{Kh}}(\tau_1(0))$ and $\widetilde{\mathrm{Kh}}(\tau_1(1))$ (left), and the groups $\widetilde{\mathrm{Kh}}(\tau_2(0))$ and $\widetilde{\mathrm{Kh}}(\tau_2(1))$ (right). The δ^+ grading for T_1 is the second column, while for T_2 it is the third.

and

$$\widetilde{\mathrm{Kh}}(\tau_2(n)) \cong \mathbb{F}^4 \oplus \mathbb{F}^{20} \oplus \mathbb{F}^{16+n}$$

for $n \geq 0$ by calculating that

$$\widetilde{\mathrm{Kh}}(\tau_1(1)) \cong \mathbb{F}^{16} \oplus \mathbb{F}^{21} \oplus \mathbb{F}^4$$

and

$$\widetilde{\mathrm{Kh}}(\tau_2(n)) \cong \mathbb{F}^4 \oplus \mathbb{F}^{20} \oplus \mathbb{F}^{17}$$

applying Lemma 5.1. As a result, while the groups clearly distinguish the links $\tau_1(n)$ and $\tau_2(n)$, we have that $\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau_i(n)) = 40 + n$ for $n \geq 0$.

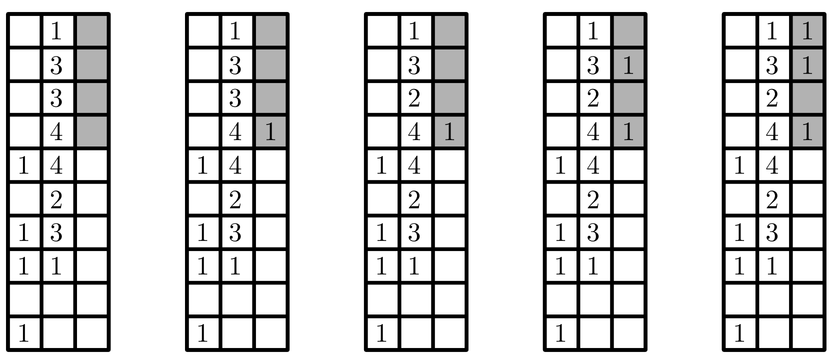


Figure 7.6 $\widetilde{\mathrm{Kh}}(\tau_2(n))$ for $n = -18, -17, -16, -15, -14$ (from left to right). The δ^+ grading is highlighted for $m = -18$ in the notation of Lemma 5.1

In fact, it can be verified that this is true for all $n > -16$, and indeed by inspection of Figure 7.6 we have that

$$\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau_i(n)) = \begin{cases} 8 - n & \text{for } n < -16 \\ 26 & \text{for } n = -16 \\ 40 + n & \text{for } n > -16 \end{cases}$$

for $i = 1, 2$ (compare Figure 6.6).

Remark 7.8. *Using the width of* $\widetilde{\mathrm{Kh}}(\tau_1(n))$ *we were able to conclude that the* $(-2, 5, 5)$-

pretzel knot does not admit finite fillings (see Theorem 6.10). Notice that such a result depends, in general, on the choice of involution, as demonstrated by this example: the same conclusion cannot be made using T_2 since $w_{\min} = 2$ in this case.

7.2.2 Paoluzzi's example

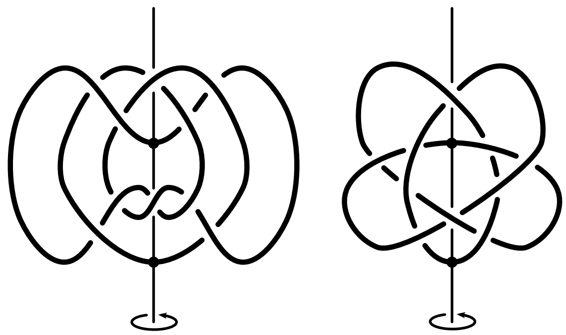


Figure 7.7 Two views of the knot 10_{155}.

The knot 10_{155} admits a pair of strong inversions as shown in Figure 7.7, however rather than meeting in a point (as in the previous example), Paoluzzi shows that the two fixed point sets for the respective involutions in this setting form a Hopf link (Paoluzzi, 2005, Section 5, Figure 10).

Proceeding as in Section 6.5, the zero surgery has two distinct branch sets. These are illustrated in Figure 7.8

Therefore, we have that

$$\widetilde{\mathrm{Kh}}(\tau_1(0)) \cong \mathbb{F}^{16} \oplus \mathbb{F}^{25} \oplus \mathbb{F}^{9}$$

and

$$\widetilde{\mathrm{Kh}}(\tau_2(0)) \cong \mathbb{F}^{9} \oplus \mathbb{F}^{21} \oplus \mathbb{F}^{12} \oplus \mathbb{F}^{4} \oplus \mathbb{F}^{4}$$

so that the ranks coincide. By considering the +5 surgery (say) in each case, and applying Lemma 5.1 we may conclude that

$$\widetilde{\mathrm{Kh}}(\tau_1(n)) \cong \mathbb{F}^{16} \oplus \mathbb{F}^{24} \oplus \mathbb{F}^{9+n}$$

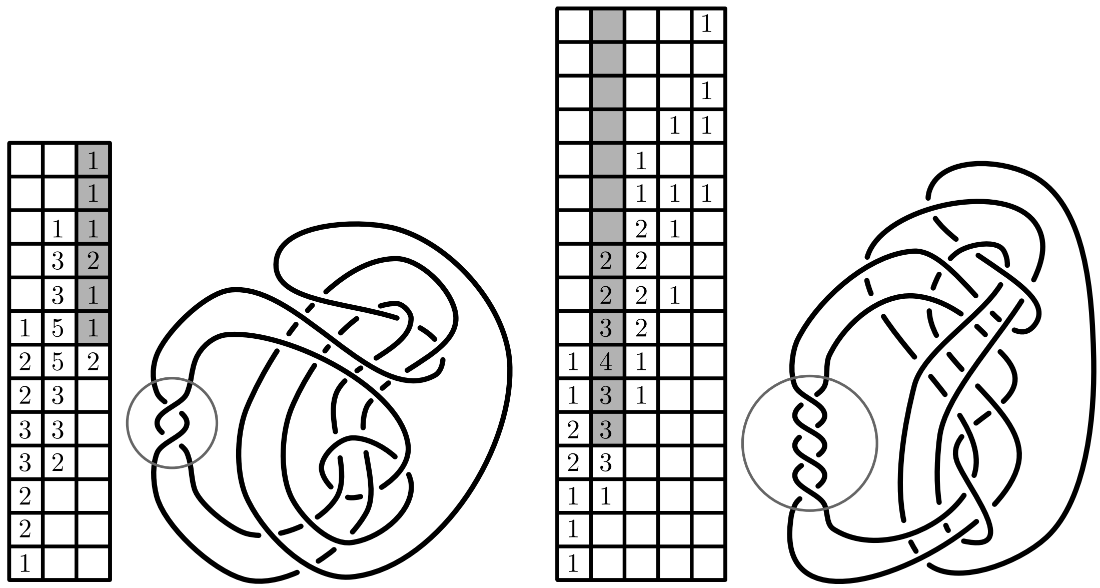


Figure 7.8 The homology of the pair of branch sets associated to the zero surgery on the knot 10_{155}. Note that the Euler characteristic (and hence the determinant) is zero in both cases.

and

$$\widetilde{\mathrm{Kh}}(\tau_2(n)) \cong \mathbb{F}^9 \oplus \mathbb{F}^{20} \oplus \mathbb{F}^{12+n} \oplus \mathbb{F}^4 \oplus \mathbb{F}^4$$

for all $n > 0$. As a result,

$$\operatorname{rk} \widetilde{\mathrm{Kh}}(\tau_i(n)) = \begin{cases} 50 & n = 0 \\ 48 + n & n > 0 \end{cases}$$

for $i = 1, 2$. Interestingly, in this case the width alone is enough to distinguish these branch sets, while the rank is not.

Remark 7.9. *We note that both branch sets give rise to generic tangles, so that in either case we may conclude that* 10_{155} *does not admit finite fillings. More generally, since*

$$\Delta_{10_{155}}(t) = -t^{-3} + 3t^{-2} - 5t^{-1} + 7 - 5t + 3t^2 - t^3$$

this knot does not admit L-space surgeries.

7.3 Manifolds branching in 3 distinct ways

By a construction of Zimmermann (Zimmermann, 1997, Section 5), the following method gives rise to a two-fold branched cover of S^3 with three distinct branch sets (this example taken from (Paoluzzi, 2005)).

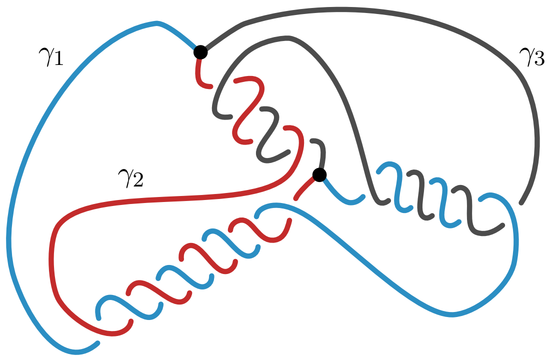


Figure 7.9 A knotted theta graph Γ.

Consider the knotted theta graph Γ shown in Figure 7.9. Notice that this graph has the property that $\Gamma \smallsetminus \gamma_i$ is the trivial knot, and as a result $S^3 = \mathbf{\Sigma}(S^3, \Gamma \smallsetminus \gamma_i)$ for each edge γ_i.

Now define $K_i = \widetilde{\gamma}_i$, the lift of arc meeting the (trivial) branch set in 2 points giving rise to a knot in S^3. That is,

$$K_i = \widetilde{\gamma}_i \hookrightarrow \mathbf{\Sigma}(S^3, \Gamma \smallsetminus \gamma_i) \cong S^3.$$

The K_i may be determined by the method of (Zimmermann, 1997, Section 5), and are given in (Paoluzzi, 2005, Figure 3).[3]

Zimmermann's result is that the collection of 3-manifolds $\left\{\mathbf{\Sigma}(S^3, K_i)\right\}$ for $i = 1, 2, 3$ are homeomorphic, as they are all obtained as a branched cover of the graph Γ.

[3]Note however that the theta graph given in (Paoluzzi, 2005, Figure 3) is incorrect.

We compute

$$\operatorname{rk}\widetilde{\mathrm{Kh}}(K_i) = 191$$

and

$$w(K_i) = 2i + 2.$$

CHAPTER VIII

DOES KHOVANOV HOMOLOGY DETECT THE TRIVIAL KNOT?

The question of the existence of a non-trivial knot with trivial Jones polynomial has received considerable attention since the discovery of this revolutionary knot invariant. While the question remains open, Khovanov homology – the categorification of the Jones polynomial – gives rise to a natural reformulation: Is there a non-trivial knot for which the reduced Khovanov homology has rank 1? This chapter explores certain aspects of this question, and in particular establishes a class of knots for which the answer is no. As a result, Khovanov homology may be used to construct combinatorial knot invariants that detect the trivial knot.

Some of the results in this chapter are joint work with M. Hedden (Hedden and Watson, 2008).

8.1 Strongly invertible knots

We begin with an observation regarding Khovanov homology and non-trivial, strongly invertible knots in S^3.

Theorem 8.1. *Let K be a strongly invertible knot in S^3 with associated quotient tangle $T=(B^3,\tau)$. Then $\widetilde{\mathrm{Kh}}(\tau(n))$ is thin for every non-zero integer n if and only if K is the trivial knot.*

Proof. If K is the trivial knot, then $\tau(n)$ is two-bridge link, and $\widetilde{\mathrm{Kh}}(\tau(n))$ is thin for

$n \neq 0$ (c.f. Theorem 4.24). We treat the converse.

Recall that from Corollary 3.13 we have the following inequalities:

$$|H_1(\mathbf{\Sigma}(S^3, L); \mathbb{Z})| \leq \operatorname{rk} \widehat{\mathrm{HF}}(\mathbf{\Sigma}(S^3, L)) \leq \operatorname{rk} \widetilde{\mathrm{Kh}}(L)$$

Further, whenever $\widetilde{\mathrm{Kh}}(L)$ is thin, $|H_1(\mathbf{\Sigma}(S^3, L); \mathbb{Z})| = \operatorname{rk} \widetilde{\mathrm{Kh}}(L)$ so that $\mathbf{\Sigma}(S^3, L)$ is an L-space (see Proposition 3.16).

Now suppose that $\widetilde{\mathrm{Kh}}(\tau(n))$ is thin for every non-zero integer n. Then from the discussion above $\Sigma^2_{\tau(n)} \cong S^3_n(K)$ is an L-space for $n \neq 0$. Applying Proposition 3.35, K must be the trivial knot. □

Using the symmetry group of the knot it is possible to determine when a knot is not strongly invertible. As a result, Khovanov homology may be used to detect the trivial knot in the following sense: since the trivial knot is strongly invertible, Khovanov homology, together with the symmetry group of the knot, detects the trivial knot via Theorem 8.1. Note that Lemma 5.1 combine to ensure that the minimal width $w_{\min}$ of $\widetilde{\mathrm{Kh}}(\tau(n))$ is determined on a finite collection of integers. However, it is certainly true that calculating the symmetry group is a difficult task in general.

In light of the relationship between Heegaard-Floer homology and Khovanov homology by way of two-fold branched covers it is interesting to recall that knot Floer homology, which is closely tied to the Heegaard-Floer homology of surgeries on a knot, detects the trivial knot (see Section 3.8). Here, Khovanov homology detects the trivial knot among knots whose complements are branched covers of tangles.

8.2 Tangle unknotting number one knots

Theorem 8.2. *(Hedden and Watson, 2008) Suppose $K \hookrightarrow S^3$ has tangle unknotting number one (as in Definition 4.1). Then $\operatorname{rk} \widetilde{\mathrm{Kh}}(K) = 1$ if and only if K is the trivial knot.*

This follows immediately from Proposition 4.10 which asserts that the two-fold branched cover of a tangle unknotting number one knot may be obtained by surgery on a knot in S^3, combined with the following more general statement.

Theorem 8.3. *Let K be a non-trivial knot in S^3 with the property that $\Sigma^2_K \cong S^3_{p/q}(K')$ for some knot K' in S^3. Then* $\operatorname{rk}\widetilde{\mathrm{Kh}}(K) > 1$.

Proof. As in Section 8.1, the proof relies heavily on the machinery of Heegaard-Floer homology, in particular Corollary 3.13 which gives the bound

$$\left|H_1(\Sigma^2_L;\mathbb{Z})\right| \leq \operatorname{rk}\widehat{\mathrm{HF}}(\Sigma^2_L) \leq \operatorname{rk}\widetilde{\mathrm{Kh}}(L)$$

Suppose that $\Sigma^2_K \cong S^3_{p/q}(K')$. By passing to the mirror image if necessary we may assume that $\frac{p}{q} > 0$ (notice that since we are considering knots the case $\frac{p}{q} = 0$ is omitted). In this setting we obtain

$$p \leq \operatorname{rk}\widehat{\mathrm{HF}}(S^3_{p/q}(K')) \leq \operatorname{rk}\widetilde{\mathrm{Kh}}(K)$$

and Theorem 8.3 follows immediately if $p > 1$. Therefore we may reduce to the case of $\frac{1}{q}$-framed surgeries so that $S^3_{1/q}(K')$ is a $\mathbb{Z}$-homology sphere. Specifically, our task is to consider the case $\operatorname{rk}\widehat{\mathrm{HF}}(S^3_{1/q}(K')) = 1$. That is, the case when surgery on a knot in S^3 yields a $\mathbb{Z}$-homology sphere L-space. But now we may apply Proposition 3.37 to conclude that K' must be the trefoil.

We are left to deal with the case when K' is the trefoil. This is a strongly invertible knot, and the associated quotient tangle is determined in Figure 4.2. The branch set associated to $+1$ surgery on K' can be identified as the $(-2,3,5)$-pretzel knot (the knot 10_{124}). Recall that this is the unique such branch set by Theorem 1.14. The result now follows by direct calculation: $\operatorname{rk}\widetilde{\mathrm{Kh}}(10_{124}) = 7$ as can be seen in Figure 2.2. □

8.3 Invariants for detecting the trivial knot

While Theorem 8.2 gives a very large class of knots on which the question of Khovanov homology detecting the trivial knot may be answered, the result becomes particularly interesting in light of the following corollary, which indicates that the Khovanov homology of many satellite knots can be used to detect the trivial knot. To describe it, let $P(K)$ be the satellite knot of K with pattern P. By pattern, we mean that P is the knot in the solid torus which is identified with the neighbourhood of K in the satellite construction.

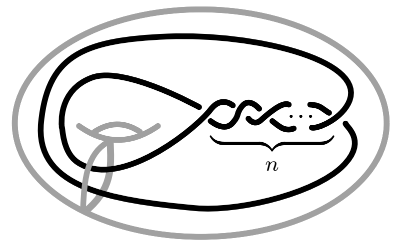
Corollary 8.4. *Let $P \hookrightarrow S^1 \times D^2$ be a knot in the solid torus. Suppose that*

- *For any K, $P(K)$ has tangle unknotting number one.*
- *$P(K) \simeq U$ if and only if $K \simeq U$, where U is the trivial knot.*

Then $\operatorname{rk} \widetilde{\mathrm{Kh}}(P(K)) = 1$ if and only if $K \simeq U$. In particular, the reduced Khovanov homology of the satellite operation defined by P detects the trivial knot.

Proof. Observe that if $K_1 \simeq K_2$ then $P(K_1) \simeq P(K_2)$, so that the operation defined by P does indeed descend to isotopy classes of knots. Given an invariant of a knot, K, this observation allows us to define infinite families of invariants: simply apply the invariant to all the various satellites of K.

In the case at hand, the invariant we are considering is the reduced Khovanov homology. Suppose that we choose a pattern P so that $P(K)$ has tangle unknotting number one for *every* knot K, and so that $P(K) \simeq U$, if and only if U is the trivial knot. In this situation, Theorem 8.2 applies to show that $\operatorname{rk} \widetilde{\mathrm{Kh}}(P(K)) = 1$ if and only if $P(K)$ is the trivial knot which, in turn, happens if and only if K is trivial. $\square$



A simple infinite family of satellite constructions whose Khovanov homologies detect the trivial knot are provided by the patterns shown in the solid torus on the right, where n denotes the number of half twists. It is straightforward to verify that each of these patterns satisfies the hypotheses of Corollary 8.4. Note that the $(2, \pm 1)$-cable of K is obtained for $n = \pm 1$. Similarly, the positive (respectively negative) clasp, untwisted Whitehead double of K is obtained for $n = 2$ (respectively $n = -2$). The case $n = 0$ is always the trivial knot, while the convention $n = \frac{1}{0}$ gives rise to the 2-cable of the knot K (this latter satellite is a link, and is handled by a different technique in (Hedden, 2008)).

As a result, we obtain an infinite family of invariants, each of which detects the trivial knot: denoting by $K_n = P(K)$ the satellite using the pattern specified by the figure, with n half-twists, we have that $\operatorname{rk} \widetilde{\mathrm{Kh}}(K_n) = 1$ if and only if K is trivial for any choice $n \neq 0$.

We remark that for the satellites specified by the figure, it is straightforward to determine the knot in S^3 on which one performs surgery to obtain the double branched covers:

$$\mathbf{\Sigma}(S^3, K_n) \cong S^3_{1/n}(K \# K)$$

Indeed, there is an obvious strong inversion on $K \# K$ exchanging the two summands. From this, one can see that the quotient is S^3 and the image of the fixed-point set is K_n. See Akbulut and Kirby (Akbulut and Kirby, 1980) or Montesinos and Whitten (Montesinos and Whitten, 1986) for details.

8.4 Khovanov homology and L-space homology spheres

We remark that the answer to a seemingly more difficult question (c.f. Question 3.19) may shed light on the question of whether Khovanov homology detects the trivial knot.

Proposition 8.5. *If the Poincaré homology sphere is the only non-trivial, prime, L-*

space, integer homology 3-sphere, then Khovanov homology detects the trivial knot.

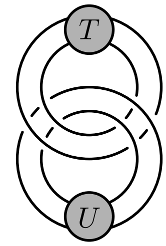
Proof. Let K be a non-trivial, prime knot in S^3. Then $\det(K)$ is non-zero, and provides a lower bound for $\operatorname{rk}\widetilde{\mathrm{Kh}}(K)$. Thus we need only consider the case when $\det(K) = 1$, that is, when $\mathbf{\Sigma}(S^3, K)$ is an integer homology sphere. More specifically, we need only consider the case when $\mathbf{\Sigma}(S^3, K)$ is an L-space, integer homology 3-sphere, since $\operatorname{rk}\widehat{\mathrm{HF}}(\mathbf{\Sigma}(S^3, K))$ provides a lower bound for $\operatorname{rk}\widetilde{\mathrm{Kh}}(K)$ (see Corollary 3.13).

If the answer to Question 3.19 is yes, then $\mathbf{\Sigma}(S^3, K)$ must be the Poincaré homology sphere. However, as we have seen, this implies that K is the knot 10_{124} with $\operatorname{rk}\widetilde{\mathrm{Kh}}(K) = 7$. As a result, Khovanov homology detects the trivial knot. □

Note that it would be enough to show that the Poincaré homology sphere is the only non-trivial, prime, L-space, integer homology 3-sphere among two-fold branched covers of S^3 to obtain the above result. However, does not appear to simplify Question 3.19 in any obvious sense.

We emphasize that *knowing* that Khovanov homology detects the trivial knot does not give any information towards Question 3.19. Indeed, this seems to be a much harder problem in general. On the other hand, an example of a non-trivial knot with trivial Khovanov homology would immediately yield a new L-space integer homology 3-sphere as two-fold branched cover.

8.5 Some examples of Eliahou, Kauffman and Thistlethwaite



The basic construction of (Eliahou et al., 2003) is that given a pair of tangles, wired together as shown on the right, there is an operation altering the diagram that is undetected by the bracket polynomial (and hence the Jones polynomial, up to a possible shift). Consider the action of the 3-strand braid group on tangles described in Section 4.4. For a given braid $\beta \in B_3$ denote the result of the action of β applied to a tangle T by T^β. For the fixed diagram on the right for the link L (for given tangles T and U), denote L^β the link obtained by replacing the pair (T, U) with the pair $(T^\beta, U^{\beta^{-1}})$. Eliahou, Kauffman and Thistlethwaite prove the following:

Proposition 8.6. *(Eliahou et al., 2003) The links L and L^β have the same Jones polynomial, up to a possible shift, for $\beta = \sigma_2^2\sigma_1^{-1}\sigma_2^2$.*

Recall that $H_1(\mathbf{\Sigma}(S^3, \mathbb{Q})) \neq 0$ if and only if $\det(L) = 0$. Some version of the following may be found in (Hedden, 2008).

Proposition 8.7. *Let L be a link with $\det(L) = 0$. If $||\phi||_T > 1$ for $[\phi] \in H_2(\mathbf{\Sigma}(S^3, \mathbb{Q}); \mathbb{Z})$ then $\operatorname{rk}\widetilde{\mathrm{Kh}}(L) > 1$. Here $||\cdot||_T$ denotes the Thurston norm.*

When T and U are rational tangles, notice that the resulting link L is composed of a pair of two-bridge links. Dunbar shows that, in the case that both of these links are non-trivial torus links, $\mathbf{\Sigma}(S^3, L)$ is geometric and has Solv geometry (Dunbar, 1988, Table 9). In particular, $\mathbf{\Sigma}(S^3, L)$ is a torus bundle so that $||\phi||_T = 1$ rendering Proposition 8.7 ineffective. In this setting however, it can be shown that for non-trivial links L, the Jones polynomial is trivial if and only if the pair of torus links are the trefoil and its mirror (Eliahou et al., 2003). That is, L is the closure of the 4-braid

1	1
1	1
2	2
3	2
2	3
2	2
1	1
1	1

$$\sigma_3^3(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})^2\sigma_1^{-3}.$$

The reduced Khovanov homology for this link is displayed in the right, hence $\operatorname{rk}\widetilde{\mathrm{Kh}}(L) = 26$ for this particular link (notice that $\chi\left(\widetilde{\mathrm{Kh}}(L)\right) = 13 - 13$ and that the bracket polynomial will be some shift of the bracket for the two-component trivial link).

Note that since Dunbar shows that the case of linked torus knots has geometric two-fold branched cover, we can conclude that the cover is geometric for any choice of rational tangles. This is particularly useful in light of Dunbar's classification: perusing the tables of (Dunbar, 1988) we conclude immediately that $\mathbf{\Sigma}(S^3, L)$ must be hyperbolic (in which case $||\phi||_T > 1$) or Seifert fibered.

The latter case is ruled out by observing that the link L is neither torus nor Montesinos whenever one of the underlying two-bridge links is hyperbolic. Note that L cannot be a torus link if one of its components is not a torus link, so it remains to show that the link L is not Montesinos. Notice however that as a satellite of the Hopf link, this possibility is ruled out.

As a result, whenever the pair of tangles (T, U) for L are rational tangles, if L is non-trivial with $\det(L) = 0$ then $\operatorname{rk} \widetilde{\mathrm{Kh}}(L) > 1$. In summary, since the particular family $LL_2(n)$ of (Eliahou et al., 2003) is contained in this family, we have:

Theorem 8.8. *The family $LL_2(n)$ with trivial Jones polynomial are distinguished from the trivial link by Khovanov homology when $n \neq 0$.*

In this notation, $LL_2(0)$ corresponds to the two-component trivial link. Note that we have proved that any non-trivial link formed by two-bridge knots, modelled on the Hopf link as constructed here, must have non-trivial Khovanov homology (compare Section 8.3).

CONCLUSION

The relationship between Khovanov homology and Heegaard-Floer homology indicates that approaching Khovanov homology by way of two-fold branched covers is natural. Perhaps more surprising, the correspondence between the complexity of the geometry in the two-fold branched cover and the coarse complexity of the Khovanov homology of the branch set (measured in terms of width) arises without reference to Heegaard-Floer homology, and suggests that further geometric properties and applications may be possible by way of Khovanov homology. These relationships – between Heegaard-Floer homology and Khovanov homology, and between Khovanov homology and the geometry of the two-fold branched cover – should be studied further. As such it seems fitting to conclude with a list of problems that may act as a guide for future work.

Strengthening the relationship to Heegaard-Floer homology

Lemma 5.1 gives a strong analogy to Heegaard-Floer homology in the context of surgery on knots in S^3 (see Remark 5.2). As a result, though the splitting

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right)$$

is a consequence of the simplicity of $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, it is natural to ask if this splitting is natural, in the following sense:

Question. *Let M be a simple, strongly invertible knot manifold, with associated quotient tangle $T = (B^3, \tau)$. Is there a choice of representative for T with the property that*

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \textstyle\bigoplus_{q=0}^{n-1}\left(\bigoplus_{\delta=1}^{k}\mathbb{F}^{b_\delta}\right)[0,q]\right)$$

for $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \bigoplus_{\delta=1}^{k}\mathbb{F}^{b_\delta}$?

We remark that, whenever $k = 1$ (so that $\tau(\frac{1}{0})$ is thin), the proof of Lemma 5.1 goes through as before. More generally, in specific instances, it is certainly possible to say something concrete about the behaviour of the width, despite the possibility differentials interacting among the $\left(\bigoplus_{\delta=1}^{k} \mathbb{F}^{b_\delta}\right)[0, i]$.

While a better understanding of this question is of interest regarding the interaction of Khovanov homology and Heegaard-Floer homology, immediate application of results in this direction would be the calculation of Khovanov homology for closures of arbitrary tangles by rational tangles, without returning to the complex level for the connecting homomorphisms. Aside from the Lee-Rasmussen spectral sequence, the skein exact sequence is currently the only computational tool in Khovanov homology.

L-space knots

Khovanov homology may be used as an obstruction to lens space surgeries and finite fillings, while Heegaard-Floer homology obstructs L-spaces, a decidedly larger class of manifolds. Call K an *L-space knot* if it admits an L-space surgery. It would be very interesting to have a classification of L-space knots in terms of Khovanov homology (at least among strongly invertible knots).

Let $K \hookrightarrow S^3$ be a strongly invertible knot, with canonical associated quotient tangle $T = (B^3, \tau)$. Say T is *stably thin* if $w(\tau(n)) = 1$ for n large enough. Note that we may assume that $n > 0$ up to taking mirrors.

Question. *If K admits an L-space surgery, is T stably thin?*

The converse obviously holds, though we have no reason beyond never having encountered phenomena to the contrary to assume that the answer should be "yes". Furthermore, we know of no examples of L-space knots that are not strongly invertible. Such an example would be very interesting, as it would yield examples of L-spaces that do not admit a strong inversion; currently there are no known examples of this phenomenon.

Khovanov homology and the geometry of two-fold branched covers

The homological width of the branch sets for small Seifert fibered spaces may be arbitrarily large, since such manifolds (Brieskorn spheres in particular) may arise as the two-fold branched cover of torus knots. On the other hand, examples of thin links with hyperbolic two-fold branched cover are easy to produce (consider large integer surgery on a hyperbolic Berge knot, for example), and as such one should not expect to obstruct hyperbolicity using the width of the branch set.

However, it seems possible that obstructions to other geometries exist.

Question. *Can width be related to other geometries?*

In particular, we expect that Euclidean and Sol geometries may arise as two-fold branched covers of links with boundable width, and intend to pursue this question further.

Finally, w and the total rank are the simplest possible invariants that one may derive from Khovanov homology. While these are certainly homological quantities (in that they cannot be recovered from the Jones polynomial in general), Khovanov homology contains a wealth of rich and interesting structure that has yet to be explained or exploited.

APPENDIX
AN EXAMPLE: SURGERY ON THE POINCARÉ SPHERE

In application of the surgery obstructions from this work, the requirement that the knot be strongly invertible seems restrictive. However, while such an involution is required, we remark that the obstructions presented may be applied in broader settings beyond knots in the three sphere. As illustration of this, we study surgery on a strongly invertible knot in the Poincaré homology sphere, Y. Dehn surgery on knots in this manifold have been considered by Tange in the context of the Berge conjecture and Question 3.19 (Tange, 2007).

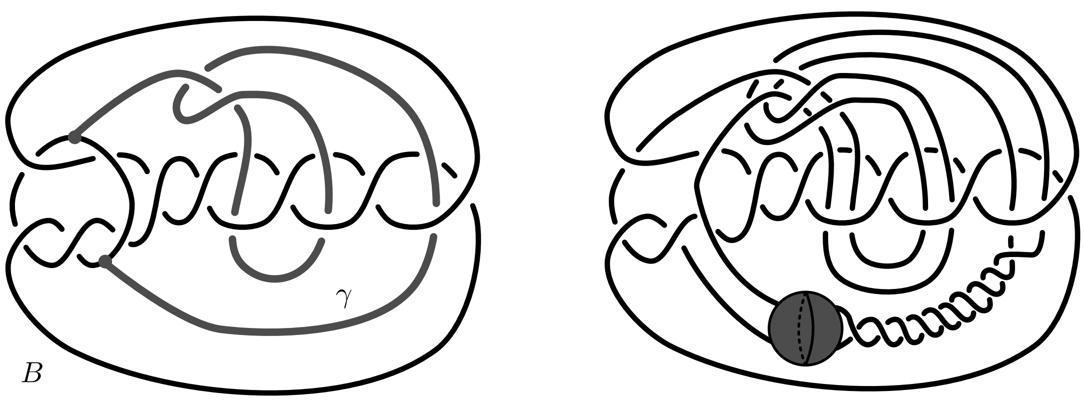


Figure 9.1 The branch set B (the knot 10_{124}) and the arc γ giving rise to $\widetilde{\gamma} = K$ in the two-fold branched cover $Y = \mathbf{\Sigma}(S^3, B)$ (the Poincaré sphere). The canonical associated quotient tangle is shown on the right. Note that $\tau(\frac{1}{0}) \simeq B$ and $\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{180} \oplus \mathbb{F}^{84}$ so that $\det(\tau(0)) = 0$.

Recall that $Y \cong \mathbf{\Sigma}(S^3, B)$ where B is the knot 10_{124}, the $(-2, 3, 5)$-pretzel. Consider the knot $K \hookrightarrow Y$ given by the lift $\widetilde{\gamma} = K$ where γ is the arc illustrated in Figure 9.1 with endpoints on the branch set B. Note that $K \hookrightarrow Y$ is strongly invertible (by construction), and that $M = Y \smallsetminus \nu(K)$ is a simple, strongly invertible knot manifold (c.f. Definition 4.6).

Since $H_1(Y;\mathbb{Z}) \cong 0$, there is a preferred longitudinal slope λ in ∂M so that $H_1(M(\lambda);\mathbb{Z}) \cong \mathbb{Z}$ and $\Delta(\mu,\lambda) = 1$. As a result, as in the case a of a knot complement in S^3, $M \cong \mathbf{\Sigma}(B^3,\tau)$ where we fix the canonical representative $T = (B^3,\tau)$ of associated quotient tangle. This tangle is illustrated in Figure 9.1; notice that $\tau(\frac{1}{0}) \simeq B$ is obtained by filling with the tangle $(B^3,)\,()$ (thus, a branch set for the trivial surgery on K) and $Y_0(K) \cong \mathbf{\Sigma}(S^3,\tau(0))$ where $\tau(0)$ is obtained by filling with $(B^3, \asymp)$.

In analysing the homology $\widetilde{\mathrm{Kh}}(\tau(\frac{p}{q}))$ for the branch sets associated to $Y_{p/q}(K)$, first recall that $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}^3 \oplus \mathbb{F}^4$ as a singly graded group (the bigraded group is illustrated on the right). As a result, we do not have a general form of stability as in Lemma 5.1, a priori. However, it will make sense to consider the groups $\widetilde{\mathrm{Kh}}(\tau(m\pm 1))$ for a fixed integer m. For example, when $m = 0$ we have that

	1
	1
1	1
1	1
1	

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau(+1)) &\cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{183} \oplus \mathbb{F}^{88}\\ \widetilde{\mathrm{Kh}}(\tau(0)) &\cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{180} \oplus \mathbb{F}^{84}\\ \widetilde{\mathrm{Kh}}(\tau(-1)) &\cong \mathbb{F}^{80} \oplus \mathbb{F}^{176} \oplus \mathbb{F}^{177} \oplus \mathbb{F}^{80}\end{aligned}$$

as relatively $\mathbb{Z}$-graded groups (which verifies in particular that $\det(\tau(0)) = 0$ and $\det(\tau(\pm 1)) = 1$, as claimed). Notice that this forces each of

$$\widetilde{\mathrm{Kh}}(\tau(0)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(-1)) \xrightarrow{0} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))\right)$$

and

$$\widetilde{\mathrm{Kh}}(\tau(+1)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(0)) \xrightarrow{0} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))\right)$$

for dimension reasons (suppressing the grading shifts), since in each case the groups in (relative) grading 3 and 4 are increased by 3 and 4 respectively.

This behaviour should not be expected in general,[1] though we do have that

$$\widetilde{\mathrm{Kh}}(\tau(m+1) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)[-\tfrac{1}{2},\tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tfrac{1}{0})[-\tfrac{1}{2}(c_\tau+m),\tfrac{1}{2}(3c_\tau+3m+2)]\right),$$

and this mapping cone may be iterated as in the proof of Lemma 5.1. For example, the groups

$$\begin{aligned}\widetilde{\mathrm{Kh}}(\tau(-11)) &\cong \mathbb{F}^{80}\oplus\mathbb{F}^{176}\oplus\mathbb{F}^{155}\oplus\mathbb{F}^{48}\\ \widetilde{\mathrm{Kh}}(\tau(-10)) &\cong \mathbb{F}^{80}\oplus\mathbb{F}^{176}\oplus\mathbb{F}^{154}\oplus\mathbb{F}^{48}\\ \widetilde{\mathrm{Kh}}(\tau(-9)) &\cong \mathbb{F}^{80}\oplus\mathbb{F}^{176}\oplus\mathbb{F}^{153}\oplus\mathbb{F}^{48}\end{aligned}$$

are illustrate in Figure 9.2. When $m=-11,-10$, these groups illustrate the behaviour of the above mapping cone. Notice that the total rank decreases by one in each case. More generally, though differentials among the $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))[x,y][0,q]$ may be present, the groups still only occupy two -fixed diagonals when $\widetilde{\mathrm{Kh}}(\tau(m+n))$ is viewed as a relatively graded group.

We now analyse the behaviour of $w(\tau(n))$ for $n\in\mathbb{Z}$. First notice that

$$\widetilde{\mathrm{Kh}}(\tau(0) \cong \mathbb{F}^{80}\oplus\mathbb{F}^{176}\oplus\mathbb{F}^{180}\oplus\mathbb{F}^{84}$$

so that

$$\widetilde{\mathrm{Kh}}(\tau(1)) \cong H_*\left(\begin{array}{cccccc} \mathbb{F}^{80} & \mathbb{F}^{176} & & \mathbb{F}^{180} & & \mathbb{F}^{84} \\ & & \overset{0}{\searrow} & & \overset{0}{\searrow} & \\ & & & \mathbb{F}^{3} & & \mathbb{F}^{4}\end{array}\right)$$

by our calculations above. More generally, for $m>0$

$$\widetilde{\mathrm{Kh}}(\tau(m+1) \cong H_*\left(\begin{array}{cccccc} \mathbb{F}^{b_1} & \mathbb{F}^{b_2} & & \mathbb{F}^{b_3} & & \mathbb{F}^{b_4} \\ & & \searrow & & \searrow & \\ & & & \mathbb{F}^{3} & & \mathbb{F}^{4}\end{array}\right)$$

---

[1] However, it is very interesting that in this particular example $\widetilde{\mathrm{Kh}}(\tau(-9+n) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(-9) \xrightarrow{0} \bigoplus_{q=0}^{n-1}\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0}))[x,y][0,q]\right)$, at least for $0<n\leq 10$, as in Lemma 5.1.

			1
			3
		1	5
		5	7
		11	8
	1	17	8
	5	22	7
	12	25	5
	19	24	3
1	25	21	1
4	29	15	
7	28	9	
10	24	4	
13	18	1	
13	10		
12	4		
10	1		
6			
3			
1			

			1
			3
		1	5
		5	7
		10	8
	1	17	8
	5	21	7
	12	25	5
	19	24	3
1	25	21	1
4	29	16	
7	28	9	
10	24	4	
13	18	1	
13	10		
12	4		
10	1		
6			
3			
1			

			1
			3
		1	5
		4	7
		10	8
	1	16	8
	5	21	7
	12	25	5
	19	24	3
1	25	22	1
4	29	16	
7	28	9	
10	24	4	
13	18	1	
13	10		
12	4		
10	1		
6			
3			
1			

Figure 9.2 The groups $\widetilde{\text{Kh}}(\tau(-11))$, $\widetilde{\text{Kh}}(\tau(-10))$ and $\widetilde{\text{Kh}}(\tau(-9))$ from left to right. The change in each group (corresponding to a +1 surgery in the cover) is circled; the support of $\widetilde{\text{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}^3 \oplus \mathbb{F}^4$ is shaded in grey so that $\widetilde{\text{Kh}}(\tau(m+1)) \cong H_*\left(\widetilde{\text{Kh}}(\tau(m)) \to \mathbb{F}^3 \oplus \mathbb{F}^4\right)$.

by analysing the grading shifts as in the proof of Lemma 5.1. In particular, $b_i > 0$ for all $m > 0$ due to the shift by 1 in the secondary grading at each step (note that $b_1 = 80$, for all m).

Similarly, notice that

$$\widetilde{\text{Kh}}(\tau(-1) \cong H_*\left(\begin{array}{cccc} & \mathbb{F}^3 \searrow & \mathbb{F}^4 \searrow & \\ \mathbb{F}^{80} & \mathbb{F}^{176} & \mathbb{F}^{180} & \mathbb{F}^{84} \end{array}\right)$$

this time by resolving the single negative terminal crossing (corresponding to the −1-

surgery in the cover). More generally, for $m < 0$

$$\widetilde{\mathrm{Kh}}(\tau(m-1) \cong H_* \left(\begin{array}{cccc} & \mathbb{F}^3 & \mathbb{F}^4 & \\ & \searrow & \searrow & \\ \mathbb{F}^{b_1} & \mathbb{F}^{b_2} & \mathbb{F}^{b_3} & \mathbb{F}^{b_4} \end{array} \right)$$

by inspection of the grading shifts as in Lemma 5.1. Analysing the groups in Figure 9.2, we see that $b_1 = 80$ as before (for any m), while b_4 is necessarily non-trivial due to the shift by -1 in the secondary grading at each step.

As a result, we conclude that $w(\tau(n)) = 4$ for every $n \in \mathbb{Z}$.

With this in hand, we may determine $w(\tau(\frac{p}{q}))$ for every $\frac{p}{q} \in \mathbb{Q}$: $w(\tau(\frac{p}{q}))$ is bounded above by 4 (proceeding as in Propostion 5.11) and bounded below by 4 (proceeding as in Proposition 5.13, since $w_{\min} = w_{\max} = 4$ in this case). Said another way, the function

$$w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$$

is constant, with value 4. As a result, applying Theorem 4.25 we conclude that $K \hookrightarrow Y$ does not admit finite fillings.

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