Surgery Obstructions from Khovanov homology

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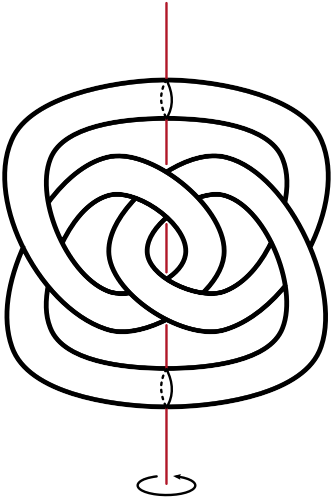
Georgia Topology Conference
May 21, 2009

Involutions and tangles

Let $K \hookrightarrow S^3$ be a strongly invertible knot.

Then there is an involution f on the knot complement $M = S^3 \setminus \nu(K)$ with one dimensional fixed point set (a pair of arcs) meeting the boundary transversely in 4 distinct points.



Involutions and tangles

Note that the quotient M/f is homeomorphic to a 3-ball.

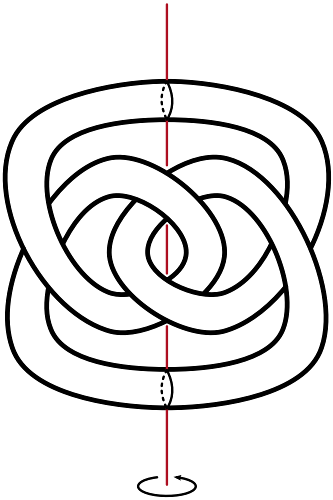
Definition

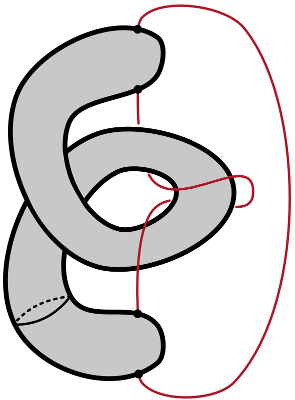
For a strongly invertible knot $K \hookrightarrow S^3$, the *associated quotient tangle* is the pair $T = (B^3, \tau)$, where τ is the image of the fixed point set of f in the quotient $M/f \cong B^3$.

As a result the knot complement is a two-fold branched cover:

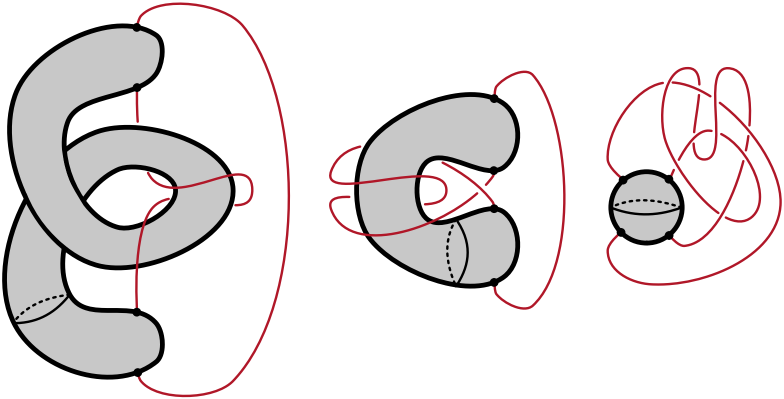
$$M \cong \mathbf{\Sigma}(B^3, \tau).$$

Example: the figure eight



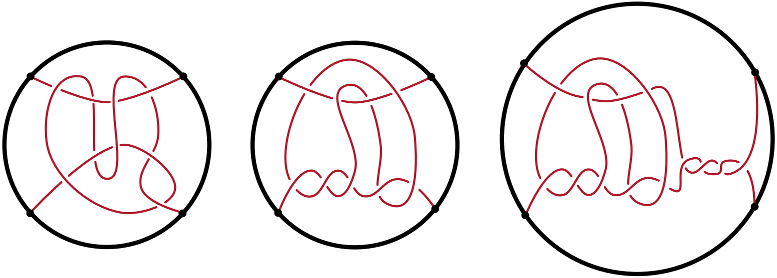


Example: the figure eight



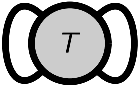
Example: the figure eight

Tangles, in this setting, are considered up to homeomorphism of the pair (B^3, τ):

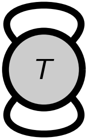


In particular, such homeomorphisms need not fix the boundary.

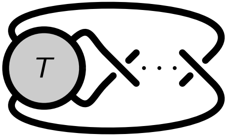
The preferred representative



By construction, the *denominator* closure of T – denoted $\tau(\frac{1}{0})$ – corresponds to the trivial surgery on K (notice that $\tau(\frac{1}{0})$ is the trivial knot).



There is a preferred choice of representative for the associated quotient tangle so that the *numerator* closure of T – denoted $\tau(0)$ – corresponds to the zero surgery in the cover: $S^3_0(K) \cong \mathbf{\Sigma}(S^3, \tau(0))$.



In particular, with this notation $\tau(n)$ is obtained by adding n half-twists so that $S^3_n(K) \cong \mathbf{\Sigma}(S^3, \tau(n))$.

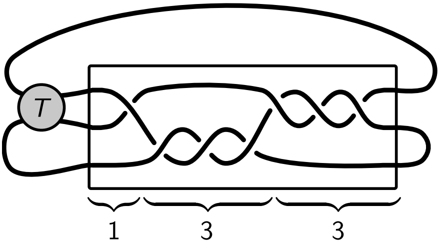
Montesinos' trick

In general,

$$S^3_{p/q}(K) \cong \mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q}))$$

where the link $\tau(\frac{p}{q})$ is obtained by attaching a rational tangle. For example:

$\tau(\frac{13}{10}) = \tau[1,3,3] =$



Branch sets for Dehn surgeries

With the observation that

$$S^3_{p/q}(K) \cong \mathbf{\Sigma}(S^3, \tau(\tfrac{p}{q}))$$

in hand, the idea is to apply the Khovanov homology of $\tau(\frac{p}{q})$ as an obstruction to exceptional Dehn surgeries on K.

Khovanov homology

The reduced Khovanov homology is a relatively $\mathbb{Z} \oplus \mathbb{Z}$-graded group $\widetilde{\mathrm{Kh}}(L)$ associated to a link $L \hookrightarrow S^3$. We work over $\mathbb{F} \cong \mathbb{Z}/2\mathbb{Z}$, with primary (cohomological) grading δ and secondary (Jones, quantum) grading q. These grading conventions are non-standard:

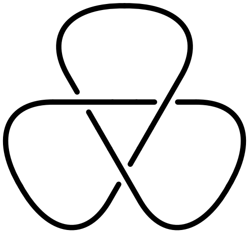
Theorem (Khovanov)

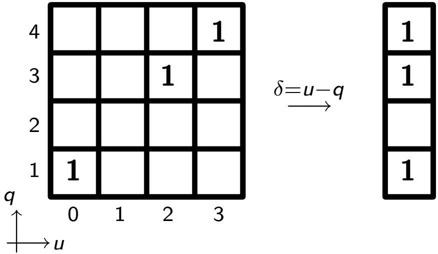
Let $u = \delta + q$, then there exists and absolute $\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$-grading on $\widetilde{\mathrm{Kh}}(L)$ so that

$$V_L(t) = \sum_{u,q} (-1)^u t^q \operatorname{rk} \widetilde{\mathrm{Kh}}^u_q(L)$$

where $V_L(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ is the Jones polynomial.

Example: the trefoil





Homological width

Definition

The homological width of a link L is given by the number of δ-gradings supporting $\widetilde{\mathrm{Kh}}(L)$. That is if

$$\bigoplus_{\delta} \widetilde{\mathrm{Kh}}^{\delta}(L) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k},$$

for $b_\delta \geq 0$ and $b_1, b_k > 0$, write $w(L) = k$.

Notice that

$$\Big| \sum_{\delta} (-1)^{\delta} \operatorname{rk} \widetilde{\mathrm{Kh}}^{\delta}(L) \Big| = \big| H_1(\boldsymbol{\Sigma}(S^3, L); \mathbb{Z}) \big|$$

since $|V_L(-1)| = \det(L) = \big| H_1(\boldsymbol{\Sigma}(S^3, L); \mathbb{Z}) \big|$.

Examples

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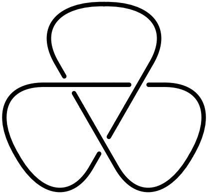
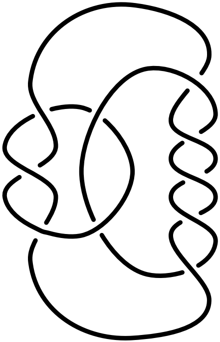
1
1



$w(3_1) = 1$

	1
	1
1	1
1	1
1	



$w(10_{124}) = 2$

Homological width

Theorem 1 (W.)

If $\mathbf{\Sigma}(S^3, L)$ *has finite fundamental group then* $w(L) \leq 2$*.*

As a first step, compare:

Theorem

If $\mathbf{\Sigma}(S^3, L)$ *is a lens space then* $w(L) = 1$*.*

Proof.

Hodgson and Rubinstein show that if $\mathbf{\Sigma}(S^3, L)$ is a lens space then L is a non-split two-bridge link; Lee proved that non-split alternating links – in particular two-bridge links – are thin. □

Manifolds with finite fundamental group

- By the orbifold theorem, having a finite fundamental group is equivalent to admitting elliptic geometry in this setting (Thurston, see Boileau-Porti).
- Manifolds with elliptic geometry are all Seifert fibered: they are either lens spaces (see previous theorem) or have base orbifold $S^2(2,2,n)$ for $n > 1$ or $S^2(2,3,n)$ for $n = 3,4,5$ (Seifert, see Scott).
- These manifolds may be constructed by considering Dehn fillings of the twisted I-bundle over the Klein bottle (base $D^2(2,2)$) or the trefoil complement (base $D^2(2,3)$) (Heil, Montesinos).
- This construction is such that the branch set in each case is recovered, and this branch set is unique (Montesinos, Boileau-Otal).

Manifolds with finite fundamental group

In summary, there exists a set of links $\mathcal{L}$ for which $L \in \mathcal{L}$ if and only if $\pi_1(\mathbf{\Sigma}(S^3, L))$ is finite.

To prove Theorem 1, we need to see that this collection of branch sets has relatively tame Khovanov homology, in the sense that $w(L) \leq 2$ whenever $L \in \mathcal{L}$.

This will rely on a particular form of **stability** enjoyed by Khovanov homology.

Surgery obstructions

Let $K \hookrightarrow S^3$ be a strongly invertible knot so that $S^3_{p/q}(K) \cong \mathbf{\Sigma}(S^3, \tau(\frac{p}{q}))$. Define

$$w_K = \min_{\frac{p}{q} \in \mathbb{Q}} \{w(\tau(\tfrac{p}{q}))\}.$$

Theorem 2 (W.)

If $w_K > 1$ then K does not admit lens space surgeries, and if $w_K > 2$ then K does not admit finite fillings. Moreover, if T is **generic** *then w_K is determined on a finite collection of integer fillings by* **stability**.

The skein exact sequence

$$\widetilde{\mathrm{Kh}}(\diagdown\!\!\!\!\diagup)$$

$$\widetilde{\mathrm{Kh}}()\,()[-\tfrac{c}{2}, \tfrac{3c+2}{2}] \nearrow \qquad \searrow \widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}, \tfrac{1}{2}]$$

$$\widetilde{\mathrm{Kh}}()\,()[-\tfrac{c}{2}, \tfrac{3c+2}{2}] \xleftarrow{[1,0]} \widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}, \tfrac{1}{2}]$$

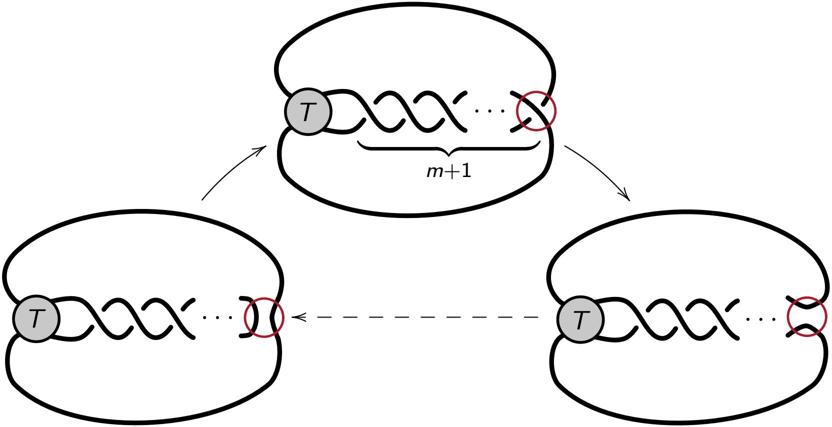
Where $c = n_-()\,() - n_-(\diagdown\!\!\!\!\diagup)$ and $\widetilde{\mathrm{Kh}}^{\delta}_{q}(L)[i,j] = \widetilde{\mathrm{Kh}}^{\delta-i}_{q-j}(L)$.

Or, as a mapping cone:

$$\widetilde{\mathrm{Kh}}(\diagdown\!\!\!\!\diagup) \cong H_*\left(\widetilde{\mathrm{Kh}}(\asymp)[-\tfrac{1}{2}, \tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}()\,()[-\tfrac{c}{2}, \tfrac{3c+2}{2}]\right)$$

A mapping cone for integer surgeries

Now when applying this to the link $\tau(m+1)$ we have:

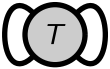


A mapping cone for integer surgeries

So that

$$\widetilde{\mathrm{Kh}}(\tau(m+1)) \cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2}, \tfrac{1}{2}] \to \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[-\tfrac{c}{2}, \tfrac{3c+2}{2}] \right)$$

where $\tau(\frac{1}{0})$ is the trivial knot and $c = c_\tau + m$ with

$c_T = n_-\Big($  $\Big) - n_-\Big($  $\Big)$

$$\widetilde{\mathrm{Kh}}(\tau(m+1))$$
$$\cong H_* \left(\widetilde{\mathrm{Kh}}(\tau(m))[-\tfrac{1}{2}, \tfrac{1}{2}] \to \mathbb{F}[-\tfrac{c_\tau}{2}, \tfrac{3c_\tau+2}{2}][0,m][-\tfrac{m}{2}, \tfrac{m}{2}] \right)$$

A mapping cone for integer surgeries

Stability Lemma

For any integer m, and positive integer n,

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \bigoplus_n \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))\right)$$

as a relatively $\mathbb{Z}\oplus\mathbb{Z}$-graded group. More precisely, there exist explicit constants x and y and an identification

$$\bigoplus_{q=0}^{n-1} \widetilde{\mathrm{Kh}}(\tau(\tfrac{1}{0}))[x,y][0,q] \cong \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$$

as graded $\mathbb{F}$-vector spaces so that

$$\widetilde{\mathrm{Kh}}(\tau(m+n)) \cong H_*\left(\widetilde{\mathrm{Kh}}(\tau(m)) \to \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]\right).$$

Example: the figure eight revisited

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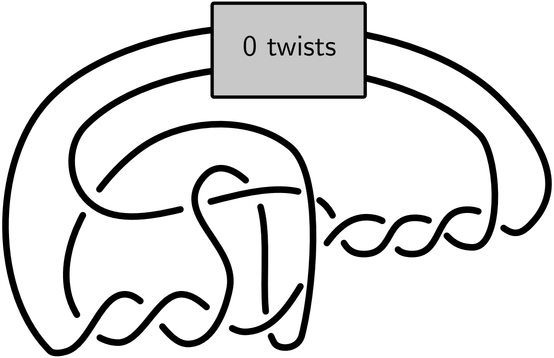
		1
	1	1
	1	1
1	1	
	1	
	1	



Example: the figure eight revisited

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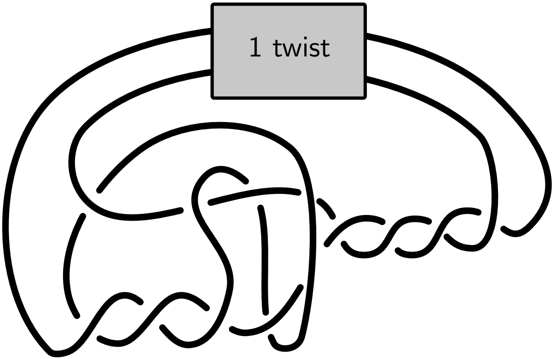
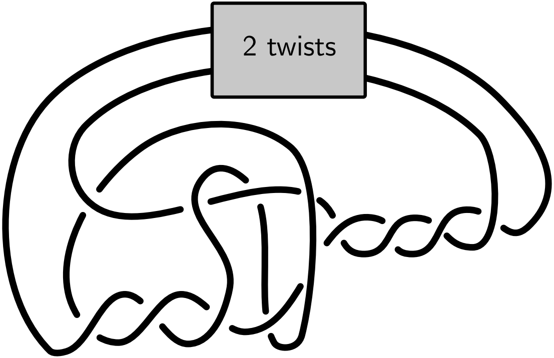


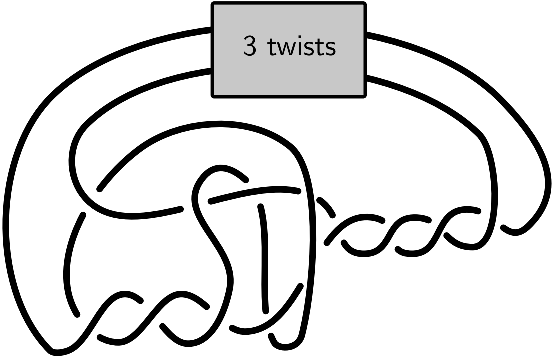
Example: the figure eight revisited

		1
		1
	1	1
	2	1
	1	
	1	
	1	



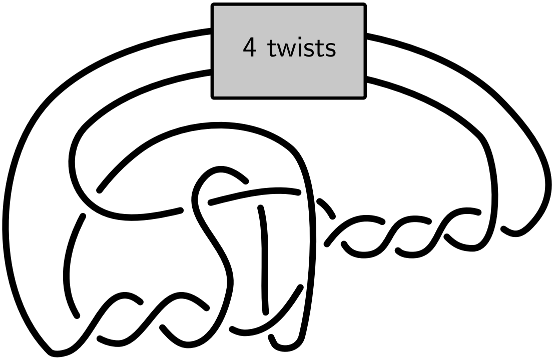
Example: the figure eight revisited

		1
		1
	1	1
	1	
	2	1
	1	
	1	
	1	



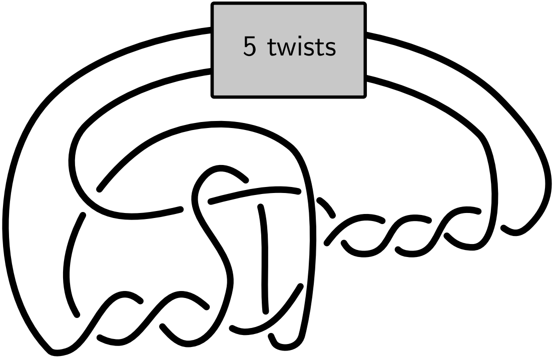
Example: the figure eight revisited

		1
		1
	2	1
	1	
	2	1
	1	
	1	
	1	



Example: the figure eight revisited

		1
	1	1
	2	1
	1	
	2	1
	1	
	1	
	1	



Example: the figure eight revisited

Notice that $w(\tau(n)) = 2$ for $n > 0$, and $w(\tau(n)) = 3$ for $n \leq 0$ as a consequence of the stability lemma.

By the cyclic surgery theorem, a lens space surgery on S^3 arises as an integer surgery.

Therefore, we recover the well known fact that the figure eight does not admit lens space surgeries:

$$\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4$$

implies that $w > 1$ for branch sets associated to integer surgeries.

Consequences of stability

Lemma

For $N >> 0$ the exact sequence for $\widetilde{\mathrm{Kh}}(\tau(N+1))$ splits so that, ignoring gradings,

$$\widetilde{\mathrm{Kh}}(\tau(N+1)) \cong \widetilde{\mathrm{Kh}}(\tau(N)) \oplus \mathbb{F}.$$

Lemma

Up to overall shift the generators $\widetilde{\mathrm{Kh}}(\tau(\frac{1}{0})) \cong \mathbb{F}$, when they survive in homology, are all supported in a single relative δ-grading.

Maximum and minimum width

Definition

For a given strongly invertible knot and preferred associated quotient tangle, define

$$w_{\max} = \max_{n \in \mathbb{Z}} \{w(\tau(n))\}$$

and

$$w_{\min} = \min_{n \in \mathbb{Z}} \{w(\tau(n))\}.$$

Lemma

Either $w_{\max} = w_{\min}$ *or* $w_{\max} = w_{\min} + 1$.

An upper bound for width

With a view to proving Theorem 1:

Proposition

Let K be a strongly invertible knot with preferred associated quotient tangle T. Then $w(\tau(\frac{p}{q})) \leq w_{\mathrm{max}}$.

σ-normalized Khovanov homology

To prove the proposition, it is natural to introduce

$$\widetilde{\mathrm{Kh}}_\sigma(L) \cong \widetilde{\mathrm{Kh}}(L)[-\tfrac{\sigma(L)}{2}]$$

as an *absolutely* $\mathbb{Z}$-graded object where $\sigma(L)$ is the signature.

Theorem (Manolescu-Ozsváth)

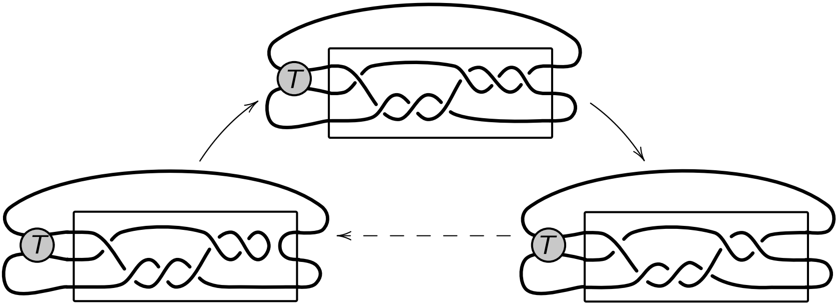
$$\widetilde{\mathrm{Kh}}_\sigma(\times) = H_*\left(\widetilde{\mathrm{Kh}}_\sigma(\asymp) \to \widetilde{\mathrm{Kh}}_\sigma()\,()\right)$$

if $\det(\asymp), \det()\,() > 0$ *and* $\det(\times) = \det(\asymp) + \det()\,()$.

It is possible to prove a variant of this statement when the determinant of one of the resolutions vanishes.

Resolutions and continued fractions

$$\frac{p}{q} = \frac{13}{10} = [1, 3, 3]$$



$$\frac{p_1}{q_1} = \frac{4}{3} = [1, 3]$$

$$\frac{p_0}{q_0} = \frac{9}{7} = [1, 3, 2]$$

$$\frac{13}{10} = \frac{4+9}{3+7}$$

Resolutions and continued fractions

In general,

$$\frac{p}{q} = \frac{p_0+p_1}{q_0+q_1}$$

when

$$\frac{p}{q} = [a_1, \dots, a_{r-1}, a_r - 1, 1] = [a_1, \dots, a_{r-1}, a_r]$$

and $\frac{p_0}{q_0}, \frac{p_1}{q_1}$ are the continued fractions

$$[a_1, \dots, a_{r-1}], [a_1, \dots, a_{r-1}, a_r - 1].$$

Since $\det(\tau(\frac{p}{q})) = |H_1(\mathbf{\Sigma}(S^3, \tau(\frac{p}{q})); \mathbb{Z})| = |H_1(S^3_{p/q}(K); \mathbb{Z})| = p$ we have that

$$\det(\tau(\tfrac{p}{q})) = \det(\tau(\tfrac{p_0}{q_0})) + \det(\tau(\tfrac{p_1}{q_1}))$$

and Manolescu and Ozsváth's theorem may be applied.

Resolutions and continued fractions

As a result, it is possible to induct in the length r of the continued fraction to prove that $w_{\max}$ is an upper bound for $w(\tau(\frac{p}{q}))$.

In particular, be successively resolving the final crossing of $\tau(\frac{p}{q}))$ it can be shown that

$$\begin{aligned} w(\tau(\tfrac{p}{q})) &\leq \max\{w(\tau\lfloor\tfrac{p}{q}\rfloor), w(\tau\lceil\tfrac{p}{q}\rceil)\} \\ &= \max\{w(\tau(a_1)), w(\tau(a_1+1))\}. \end{aligned}$$

where $\frac{p}{q} = [a_1, \ldots, a_{r-1}, a_r]$.

On Quasi-alternating links

Definition

The set of quasi-alternating links $\mathcal{Q}$ is the smallest set of such that:

- The trivial knot is an element of $\mathcal{Q}$, and
- if L admits a projection with distinguished crossing  for which each resolution gives an element of $\mathcal{Q}$, and

det() = det() + det(),

then $L \in \mathcal{Q}$ as well.

Theorem (Manolescu-Ozsváth)

Quasi-alternating links are homologically thin.

On Quasi-alternating links

Proposition

Suppose $S^3_{p/q}(K) \cong \mathbf{\Sigma}(S^3, \tau(\frac{p}{q}))$ and $\tau(N)$ is quasi-alternating for some $N > 0$. Then $\tau(\frac{p}{q})$ is quasi-alternating for all $\frac{p}{q} \geq N$.

Corollary

For large surgery on the trefoil, $\tau(\frac{p}{q})$ is quasi-alternating. In particular, $w(\tau(\frac{p}{q})) = 1$ for $\frac{p}{q} \geq 5$.

$w \leq 2$ for manifolds with finite fundamental group

Since $w_{\max} = w_{\min} + 1 = 2$ for the tangle associated to the trefoil, $w(L) \leq 2$ for $\mathbf{\Sigma}(S^3, L)$ Seifert fibered with base orbifold $S^2(2, 3, n)$.

A similar argument holds for branch sets associated to fillings of the twisted I-bundle over the Klein bottle to obtain the $S^2(2, 2, n)$ family.

This proves Theorem 1.

A lower bound for width

The proof of Theorem 2 depends on similar arguments to establish $w_{\min}$ as a lower bound for $w(\tau(\frac{p}{q}))$.

Proposition

Let K be a strongly invertible knot with **generic** *preferred associated quotient tangle T. Then $w(\tau(\frac{p}{q})) \geq w_{\min}$.*

In particular:

$$w_K = w_{\min}$$

Genericity

A tangle T is generic if either

- $w_{\max} = w_{\min}$, or
- if $b_k > 1$ where

$$\widetilde{\mathrm{Kh}}(\tau(\ell)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

and

$$\widetilde{\mathrm{Kh}}(\tau(\ell+1)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k} \oplus \mathbb{F},$$

or

- if $b_1 > 1$ where

$$\widetilde{\mathrm{Kh}}(\tau(\ell)) \cong \mathbb{F} \oplus \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}$$

and

$$\widetilde{\mathrm{Kh}}(\tau(\ell+1)) \cong \mathbb{F}^{b_1} \oplus \cdots \oplus \mathbb{F}^{b_k}.$$

Genericity

For example, for the figure eight we had that

$$\widetilde{\mathrm{Kh}}(\tau(0)) \cong \mathbb{F} \oplus \mathbb{F}^5 \oplus \mathbb{F}^4$$

and

$$\widetilde{\mathrm{Kh}}(\tau(+1)) \cong \mathbb{F}^5 \oplus \mathbb{F}^4$$

so that the width *decays* but $b_1 = 5$ so the tangle is generic. Since the figure eight is amphicheiral, we recover:

Theorem (Thurston)

The figure eight does not admit finite fillings.

A lower bound for width

Suppose that $w_{\text{min}} = w_{\text{max}} = w$. Then as before

$$\widetilde{\text{Kh}}_\sigma(\tau(\tfrac{p}{q})) \cong H_* \left(\widetilde{\text{Kh}}_\sigma(\tau(\tfrac{p_0}{q_0})) \to \widetilde{\text{Kh}}_\sigma(\tau(\tfrac{p_1}{q_1})) \right).$$

Recall that the connecting homomorphism raises δ-grading by one:

$$\widetilde{\text{Kh}}(\tau(\tfrac{p}{q})) \cong H_* \begin{pmatrix} \mathbb{F}^{b_1} & & \mathbb{F}^{b_2} & & \cdots & & \mathbb{F}^{b_w} \\ & \searrow & & \searrow & & \searrow & \\ \mathbb{F}^{b'_1} & & \mathbb{F}^{b'_2} & & \cdots & & \mathbb{F}^{b'_w} \end{pmatrix}$$

By induction in the length of the continued fraction for $\frac{p}{q}$, $w(\tau(\frac{p}{q})) = w$.

Determining widths

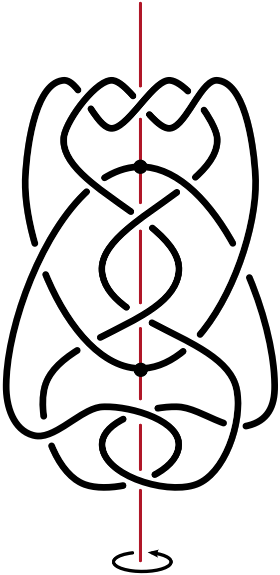
Notice that $w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$ is constant when $w_{\text{min}} = w_{\text{max}}$.

After a slightly modified argument when $w_{\text{max}} = w_{\text{min}} + 1$, we have $w(\tau(-)) : \mathbb{Q} \to \mathbb{N}$ takes values $\{w_{\text{min}}, w_{\text{max}}\}$ in the generic setting.

This proves Theorem 2: for generic tangles, the minimum width is determined on the integer fillings. That is,

$$w_K = w_{\text{min}}.$$

Example: the knot 14^n_{11893}



Theorem (Ozsváth-Szabó)

If $K \hookrightarrow S^3$ admits an L-space surgery then

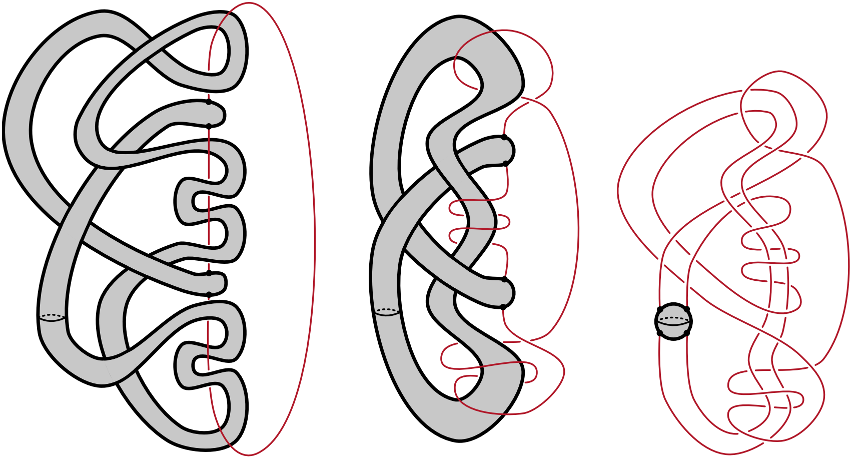
$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{-n_j} + t^{n_j})$$

for $0 < n_1 < n_2 < \cdots < n_k$.

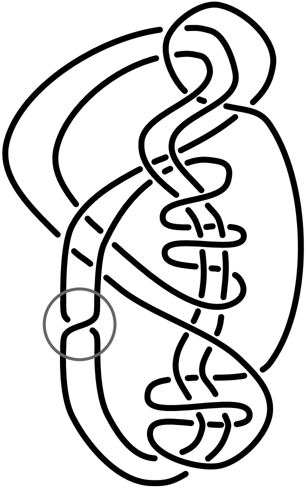
For example, $\Delta_{4_1}(t) = -t^{-1} + 3 - t$.

On the other hand,
$\Delta_{14^n_{11893}}(t) = t^{-3} - t^{-2} + t^{-1} - 1 + t - t^2 + t^3$.

Example: the knot 14^n_{11893}



			1
			2
			2
		2	3
		4	3
		5	2
	2	7	2
	4	7	1
	5	6	
1	6	5	
2	7	3	
3	5		
3	4		
4	3		
3			
2			
2			



For any n

$$w(\tau(n)) \geq 4$$

so that

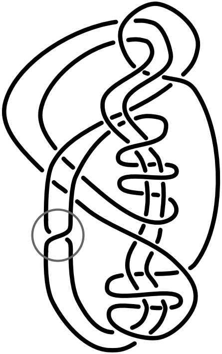
$$w_k = 4$$

Theorem

14^n_{11893} *does not admit finite fillings.*

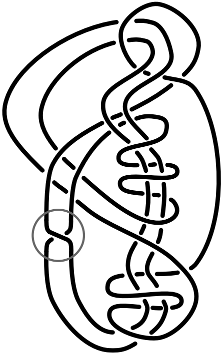
			1
			2
			2
		2	3
		4	3
		5	2
	2	7	2
	4	7	1
	5	6	
1	6	5	
2	7	3	
3	5		
3	4		
4	3		
3			
2			
2			

$\det(\tau(m+2)) = 7$



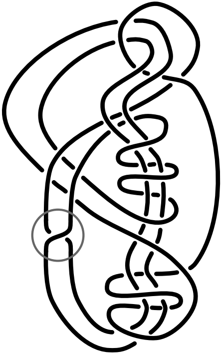
			1
			2
			2
		2	3
		4	3
		5	2
	2	7	2
	4	7	1
	5	6	
1	6	5	
2	7	④	
3	5	①	
3	4		
4	3		
3			
2			
2			

$\det(\tau(m)) = 9$



			1
			2
			2
		2	3
		4	3
		5	2
	2	7	2
	4	7	1
	5	6	
1	6	5	
2	7	3	
3	5		
3	4		
4	3		
3			
2			
2			

$\widetilde{\mathsf{Kh}}(\tau(-7)) \cong$
$\mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{39} \oplus \mathbb{F}^{16}$



			1
			2
			2
		2	3
		4	3
		5	2
	2	7	2
	4	7	1
	5	6	
1	6	5	
2	7	④	
3	5	①	
3	4		
4	3		
3			
2			
2			

$\widetilde{\mathsf{Kh}}(\tau(-9)) \cong$
$\mathbb{F}^{20} \oplus \mathbb{F}^{36} \oplus \mathbb{F}^{41} \oplus \mathbb{F}^{16}$